KAKEYA-NIKODYM PROBLEMS AND GEODESIC RESTRICTION ESTIMATES FOR EIGENFUNCTIONS

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Abstract

We record work done by the author [29] on the Kakeya-Nikodym problems, and we also record the joint work done by the author and Cheng Zhang [30] on improved geodesic restriction estimates for eigenfunctions on compact Riemannian surfaces with nonpositive curvature.

The Kakeya-Nikodym problems are among the central topics in modern Harmonic analysis. The work of the author [29] gives an alternative proof for the classical bound of Wolff for the Kakeya-Nikodym type maximal operators in Euclidean spaces $\mathbb{R}^d$, $d \geq 3$, without appealing to the induction on scales arguments.

As a consequence of the new proof, it is also shown in [29] that the same $L^{(d+2)/2}$ bound holds for Nikodym maximal function for any manifold (M^d, g) with constant curvature, which generalizes Sogge's results [22] for $d = 3$ to any $d \geq 3$. As in the 3-dimensional case, we can handle manifolds of constant curvature due to the fact that, in this case, two intersecting geodesics uniquely determine a 2-dimensional totally geodesic submanifold, which allows the use of the auxiliary maximal function to reduce the problem to a 2-dimensional one.

In the joint work of the author and Cheng Zhang [30], we prove improved L^4 geodesic restriction estimates for eigenfunctions on compact Riemannian surfaces with nonpositive curvature. We achieve this by adapting Sogge's strategy in [24]. This result improves the L^4 restriction estimate of Burq, Gérard and Tzvetkov [7] and Hu [13] by a power of $(\log \log \lambda)^{-1}$. Moreover, in the special case of compact hyperbolic surfaces, we obtain further improvements in terms of $(\log \lambda)^{-1}$ by applying the ideas from [9] and [4]. We are able to compute various constants that appeared in [9] explicitly, by lifting calculations to the universal cover $\mathbb{H}^2$.

READERS: Professor Christopher D. Sogge (Advisor), Professor Benjamin Dodson

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1.1 Introduction

The original Kakeya problem, proposed by Kakeya [14] in 1917, is to determine the minimal area needed to continuously rotate a unit line segment in the plane by 180 degrees. In 1928, Besicovitch [3] showed that such sets may have arbitrarily small measure. Moreover, Besicovitch's work indicates the existence of measure zero subsets of $\mathbb{R}^d$ which contain a unit line segment in every direction. Such sets are called Besicovitch sets or Kakeya sets.

It was later found that Kakeya sets are closely related to many fundamental problems in harmonic analysis. Fefferman [12] was the first to apply the construction of measure zero Kakeya sets to a problem of Fourier transform, namely the ball multiplier problem. It turns out that many problems in analysis require more detailed information about the size of Kakeya sets, and in particular, the fractal dimension. The Kakeya set conjecture asserts that even though the measure of a Kakeya set can be zero, it still needs to be large in the sense of fractal dimension.

Conjecture 1 (Kakeya Set Conjecture)**.** *Kakeya sets in $\mathbb{R}^d$ must have full Hausdorff/Minkowski dimension.*

There is also a stronger formulation of the conjecture in terms of maximal functions, which is called the maximal Kakeya conjecture, or the Kakeya maximal function conjecture.

Conjecture 2 (Kakeya Maximal Function Conjecture)**.** *For any $0 < \delta < 1$, given $\epsilon > 0$, there exists* *a constant C_ϵ such that*

$$\|f_\delta^*\|_{L^d(S^{d-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^d(\mathbb{R}^d)}. \tag{1.1.1}$$

Here $f_\delta^* : S^{d-1} \to \mathbb{R}$ is the Kakeya maximal function defined by:

$$f_\delta^*(\xi) = \sup_{a \in \mathbb{R}^d} \frac{1}{|T_\xi^\delta(a)|} \int_{T_\xi^\delta(a)} |f(y)|\, dy,$$

where $T_\xi^\delta(a)$ is an $1 \times \delta \times \cdots \times \delta$ tube centered at $a \in \mathbb{R}^d$ with direction $\xi \in S^{d-1}$.

Interpolating between (1.1.1) and the trivial $L^1 \to L^\infty$ estimate, one sees that natural partial results to Conjecture 2 would be the following estimate:

$$\|f_\delta^*\|_{L^q(S^{d-1})} \leq C_\epsilon \delta^{1-\frac{d}{p}-\epsilon} \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.1.2}$$

where $1 < p < d$, and $q = (d-1)p'$ are fixed. Indeed, it is well-known that an estimate like (1.1.2) for a given p would imply that Kakeya sets have Hausdorff/Minkowski dimension at least p.

For the case $d = p = 2$, Conjecture 2 was fully solved by Córdoba [10]. However, it is still open for any $d \geq 3$. When $p = (d+1)/2$, $q = (d-1)p' = d+1$, (1.1.2) follows from Drury's work on X-ray transformations [11] in 1983. In 1991, Bourgain [5] improved this result for each $d \geq 3$ to some $p(d) \in ((d+1)/2, (d+2)/2)$ by the so-called bush argument. Bourgain studied the "bush" structure where a large number of tubes intersect at a given point. Four years later, Wolff [28] generalized Bourgain's bush argument to the more refined "hairbrush argument", by considering tubes with lots of "bushes" on them. Combining the hairbrush argument and an induction on scales argument, Wolff showed that (1.1.2) holds for all $d \geq 3$, $p = (d+2)/2$. Wolff's result is still the best for Conjecture 2 when $d \leq 8$. Improved bounds have been proven in the higher dimensional cases, and for the weaker Conjecture 1 in lower dimensional cases, see e.g. [6], [15], [16].

The induction on scales argument introduced by Wolff has been an essential technique for proving such maximal inequalities. To be more specific, one can discretize Conjecture 2 by looking at the corresponding restricted weak type bound.

Conjecture 3 (Maximal Kakeya Conjecture, discrete version)**.** *Let $0 < \delta, \lambda < 1$, $1 \leq p \leq d$, and $\{T_1, \ldots, T_M\}$ be a collection of $1 \times \delta \times \ldots \times \delta$ tubes oriented in a δ-separated set of directions. For each $1 \leq i \leq M$, let $E_i \subset T_i$ be a set with $|E_i| \geq \lambda |T_i|$. Then*

$$\left| \bigcup_{i=1}^M E_i \right| \geq C_\epsilon (M\delta^{d-1}) \lambda^d \delta^{d-p+\epsilon}.$$

Remark: The Minkowski dimension version of Conjecture 1 corresponds to the case where $\lambda = 1$, and the Hausdorff dimension version essentially corresponds to the case where $\lambda \geq 1/(\log^2(1/\delta))$. Thus, while the Kakeya set conjecture is concerned with how small one can make union of tubes T_i, Conjecture 2 is concerned with how small one can make union of (possibly very small) density λ portions E_i of tubes T_i.

Wolff's induction on scales argument is also often called the "two-ends reduction" (see [27] for a detailed discussion), because it allows one to avoid the situations where each E_i is concentrated only in some small portion of the tube. That is, by two-ends reduction, it suffices to only consider portions E_i which occupy both ends of the tube in some sense. This reduction exploits the approximate scale-invariance of the Euclidean Kakeya problem, and has become a standard technique in similar problems.

It is tempting to remove such a technical argument. In 1999, Sogge [22] managed to avoid the two-ends reduction in his work on the closely related Nikodym maximal functions in 3-dimensional manifolds with constant curvature. Sogge's idea was to use a modified hairbrush argument and an optimal bound for an auxiliary maximal function. Following Sogge's idea, in 2014, Miao, Yang and Zheng [17] were able to recover Wolff's result for Kakeya maximal functions in $\mathbb{R}^3$ without the use of the two-ends reduction. In fact, Miao, Yang and Zheng also tried to recover Wolff's results for all dimension $d \geq 3$, but it seemed impossible to extend the same argument to higher dimension $d > 3$, due to the fact that their auxiliary maximal function bound involves a $\delta^{-(d-3)/2}$ loss.

The recent work [29] by the author addresses this problem. By using a more natural auxiliary maximal function and taking into account certain geometric observations, an optimal auxiliary maximal function bound was obtained. This leads to a new proof of Wolff's Kakeya maximal function bounds for all dimension $d \geq 3$, without the use of the induction on scales argument.

Theorem 1 (Xi [29])**.** *It can be shown without the induction on scales argument that the Kakeya maximal function in $\mathbb{R}^d$ satisfies*

$$\|f_\delta^*\|_{L^{\frac{(d-1)(d+2)}{d}}(S^{d-1})} \leq C_\epsilon \delta^{1-\frac{2d}{d+2}-\epsilon} \|f\|_{L^{\frac{d+2}{2}}(\mathbb{R}^d)}. \tag{1.1.3}$$

This new proof shows that Wolff's $L^{(d+2)/2}$ bounds of the Kakeya maximal function follows directly from some geometric combinatorics and Córdoba's optimal bounds for the 2-dimensional case. On one hand, it opens up a new route to get Wolff's bounds where different values of λ and different dimensions can be handled in the same way. On the other hand, since we now know how to avoid the rescaling argument, it is easier to apply similar ideas to the non-Euclidean case for

Nikodym problems following arguments in [22].

Nikodym problems are close cousins to the Kakeya problems. The Nikodym set problem is concerned with the fractal dimension of the so-called Nikodym sets. Similar to the Kakeya problems, the conjectured dimension bound for the Nikodym sets follows from a $L^d \to L^d$ bound for the corresponding Nikodym maximal function.

Recall that the Nikodym maximal function f_δ^{**} in $\mathbb{R}^d$ is defined by:

$$f_\delta^{**}(x) = \sup_{\gamma_x \ni x} \frac{1}{|T_{\gamma_x}^\delta|} \int_{T_{\gamma_x}^\delta} |f(y)|\, dy, \tag{1.1.4}$$

where the supremum runs through all the unit line segments γ_x that contains the point x. Correspondingly, we have the Nikodym maximal function conjecture.

Conjecture 4 (Nikodym Maximal Function Conjecture)**.** *For any $0 < \delta < 1$, given $\epsilon > 0$ then there exists a constant C_ϵ such that*

$$\|f_\delta^{**}\|_{L^d(\mathbb{R}^d)} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^d(\mathbb{R}^d)}. \tag{1.1.5}$$

Wolff's hairbrush argument [28] applies equally well to the Nikodym maximal function, so we have similar bounds:

$$\|f_\delta^{**}\|_{L^{\frac{(d-1)(d+2)}{d}}(\mathbb{R}^d)} \leq C_\epsilon \delta^{1-\frac{2d}{d+2}-\epsilon} \|f\|_{L^{\frac{d+2}{2}}(\mathbb{R}^d)}. \tag{1.1.6}$$

Indeed, Tao [26] showed that Kakeya maximal function conjecture is equivalent to Nikodym maximal function conjecture in Euclidean space, and furthermore, any bound like (1.1.2) is equivalent to the corresponding bound for the Nikodym maximal function.

Even though Kakeya problems and Nikodym problems are equivalent in Euclidean spaces, Kakeya problems are not natural on general manifolds since there is no unique way to identify directions at different points on a general manifold. However, we can naturally extend the definition of the Nikodym maximal function (1.1.4) to any Riemannian manifold (M, g), by replacing γ_x by any geodesic segment that contains x with length $\alpha < \min\{1, \frac{1}{2}\mathrm{Inj}(M)\}$ fixed.

In 1997, Minicozzi and Sogge [18] were the first to study the Nikodym maximal functions on general manifolds. By using a modified bush argument, they showed that for a general manifold Drury's bounds for $p = (d + 1)/2$ still hold. On the other hand, they noticed that Bourgain and Wolff's arguments relied heavily on reducing to lower dimensional subspaces. So, to extend these arguments to a manifold, one would need the existence of many totally geodesic submanifolds.

Unfortunately, for generic manifolds, this is rarely the case. Minicozzi and Sogge were able to build counter-examples by exploiting this fact. They showed that for each d there exists a manifold (M^d, g), such that this estimate breaks down if $p > \lceil (d+1)/2 \rceil$. In other words, Drury's result is the best possible at least for odd dimensional manifolds.

Clearly, if one wants to generalize Wolff's Hairbrush argument to manifolds, some additional assumptions are needed. In the later work of Sogge on Nikodym sets in 3-dimensional manifolds [22], Sogge noticed that if M has constant curvature, all 2-planes in the geodesic normal coordinates about a point are totally geodesic. Thus it seemed possible to generalize Wolff's hairbrush argument to manifolds with constant curvature. However, there was one obstacle on the way. The induction on scales argument Wolff had used seemed hard to generalize to the non-Euclidean setting. By introducing a weighted auxiliary maximal function and a more precise multiplicity argument, Sogge was able to avoid the induction on scales argument and proved the $L^{5/2}$-bounds for the Nikodym maximal function in the 3-dimensional constant curvature case.

As an application of the proof for Theorem 1, it is shown in the second part of [29] that if one works with a more natural auxiliary maximal function, Sogge's idea for 3-dimensional manifolds with constant curvature actually works for all dimensions $d \geq 3$.

Theorem 2 (Xi [29])**.** *For any $d \geq 3$, assume that (M^d, g) has constant curvature. Then for f supported in a compact subset K of a coordinate patch and all $\epsilon > 0$,*

$$\|f_\delta^{**}\|_{L^{\frac{(d-1)(d+2)}{d}}(M^d)} \leq C_\epsilon \delta^{1-\frac{2d}{d+2}-\epsilon} \|f\|_{L^{\frac{d+2}{2}}(M^d)}. \tag{1.1.7}$$

We remark that just like the Kakeya problem in Euclidean space, the Nikodym problem here is a local problem, so Theorem 2 implies the more general case (without support assumption on f). Thus it is easy to see that Theorem 2 implies Wolff's result for Nikodym maximal function in $\mathbb{R}^d$ [28] as a special 0 curvature case. Also, Theorem 2 generalizes Sogge's [22] result for 3-dimensional manifolds to any dimension higher than 3.

This chapter is organized as the following. In the next section, we modify Sogge's strategy to show that if we add in some more geometric observations, we can get rid of the $\delta^{-(d-3)/2}$ loss for the auxiliary maximal function in [17], which allows us to reduce to Cordoba's [10] optimal L^2 estimate for 2-planes. This modification helps us to recover Wolff's result. In the third section, we adapt the same idea to the Nikodym-type maximal function in the constant curvature case, and extend Sogge's result [22] to any dimension $d \geq 3$, where we shall of course need a curved version of the optimal L^2 estimate for Nikodym maximal function which is due to Mockenhaupt, Seeger and Sogge

[19].

1.2 Kakeya maximal function in Euclidean space

In this section, we prove Theorem 1. We shall follow the strategy in [22] and [17] closely, and add in some key observations. Throughout this section, we use C, c to denote various constants that only depend on the dimension.

It is well-known that it suffices to prove the following restricted weak type estimate:

$$|\{\xi \in S^{d-1} : (\chi_E)^*_\delta(\xi) \geq \lambda\}| \lesssim_\epsilon (\lambda^{-p}\delta^{p-d}|E|)^{\frac{q}{p}}, \tag{1.2.1}$$

where E is contained in the unit ball, χ_E denotes its characteristic function, $p = \frac{d+2}{2}$ and $q = \frac{(d-1)p}{p-1}$. For the sake of simplicity, we use the notation $A \lesssim_\epsilon B$ throughout the chapter to denote $A \leq C_\epsilon \delta^{-\epsilon} B$. Similarly, $B \gtrsim_\epsilon A$ means $B \geq c_\epsilon \delta^\epsilon A$.

We start by doing some standard reductions (see e.g. [5]). First, without loss of generality, we can assume that any ξ^1, $\xi^2 \in \{\xi \in S^{d-1} : (\chi_E)^*_\delta(\xi) \geq \lambda\}$ have angle $\angle(\xi^1, \xi^2) \leq 1$. Second, we take a maximal δ-separated subset $\{\xi^i\}_{i=1}^M$ of $\{\xi \in S^{d-1} : (\chi_E)^*_\delta(\xi) \geq \lambda\}$, then (1.2.1) is equivalent to

$$M\delta^{d-1} \lesssim_\epsilon (\lambda^{-p}\delta^{p-d}|E|)^{\frac{q}{p}}, \tag{1.2.2}$$

which is equivalent to

$$|E|^2 \gtrsim_\epsilon \lambda^{d+2}\delta^{d-2}(M\delta^{d-1})^{\frac{d}{d-1}}. \tag{1.2.3}$$

For each ξ^i, there is a tube $T^\delta_{\xi^i} := T^\delta_i$ satisfying

$$|E \cap T^\delta_i| \geq \lambda |T^\delta_i|. \tag{1.2.4}$$

Remark: Indeed, we will always assume $\lambda \geq \delta$ in proving (1.2.3), for the reason that in the case $\lambda \leq \delta$, it's trivial that $|E|^2 \geq |E \cap T^\delta_i|^2 \geq \lambda^2\delta^{2d-2} \geq \lambda^{d+2}\delta^{d-2} \gtrsim \lambda^{d+2}\delta^{d-2}(M\delta^{d-1})^{\frac{d}{d-1}}$. The last inequality follows from the simple fact $M\delta^{d-1} \lesssim 1$.

We start our proof by applying a multiplicity argument to these tubes, which was first introduced by Wolff. We will be using a strengthened version developed by Sogge, see Lemma 2.5 in [22]. This modification by Sogge is crucial if one wants to avoid induction on scales.

1.2.1 Multiplicity argument

Consider parameters $\theta \in [\delta, 1]$, $\sigma \in [\lambda\delta, 1]$. First, for $1 \leq j \leq M$ and $x \in T_j^\delta$ fixed, let

$$\mathfrak{L}_\theta(x, j) = \{i : x \in T_i^\delta, \angle(T_j^\delta, T_i^\delta) \in [\theta/2, \theta)\}$$

index the tubes T_i^δ containing x which intersect the fixed tube T_j^δ at angle comparable to θ. Next, let

$$\mathfrak{L}_\sigma(x, j) = \{i : x \in T_i^\delta, |T_i^\delta \cap \{y \in E : \mathrm{dist}(y, \gamma_j) \in [\sigma/2, \sigma)\}| \geq (2\log_2 \frac{1}{\delta^2})^{-1}\lambda|T_i^\delta|\}$$

index the tubes T_i^δ containing x which intersect the fixed tube T_j^δ at x such that there is a non-trivial portion of $T_i^\delta \cap E$ that has distance to the central axis of T_j^δ, γ_j, comparable to σ. Now let

$$\mathfrak{L}_{\theta,\sigma}(x, j) = \mathfrak{L}_\theta(x, j) \cap \mathfrak{L}_\sigma(x, j),$$

then we have the following

Lemma 1. *There exist $N \in \mathbb{N}$ and $\theta \in [\delta, 1]$, $\sigma \in [\lambda\delta, 1]$ that fulfill the following two cases*

I. (*Low multiplicity case*) *There are at least $M/2$ values of j for which*

$$\left|\{x \in T_j^\delta \cap E : \#\{i : x \in T_i^\delta\} \leq N\}\right| \geq \frac{\lambda}{2}|T_j^\delta|.$$

$\mathrm{II}_{\theta,\sigma}$. (*High multiplicity case at angle θ and distance σ*) *There are at least $M/(2(\log_2 1/\delta^2))^2$ values of j for which*

$$\left|\left\{x \in T_j^\delta \cap E : \#\mathfrak{L}_{\theta,\sigma}(x, j) \geq \frac{N}{(2\log_2 \frac{1}{\delta^2})^2}\right\}\right| \geq \frac{\lambda}{(4\log_2 \frac{1}{\delta^2})^2}|T_j^\delta|. \tag{1.2.5}$$

Proof. Choose the smallest $N \in \mathbb{N}$ that satisfies the low multiplicity case I. Then there must be $M/2$ values of j such that

$$|\{x \in T_j^\delta \cap E : \#\{i : x \in T_i^\delta\} \geq N/2\}| \geq \frac{\lambda}{2}|T_j^\delta|. \tag{1.2.6}$$

We claim that for any such fixed j and $x \in T_j^\delta \cap E$ with $\#\{i : x \in T_i^\delta\} \geq N/2$ we can find $1 \leq m_{x,j} \leq \log_2 \frac{1}{\delta}$, and $1 \leq n_{x,j} \leq \log_2 \frac{1}{\lambda\delta} \leq \log_2 \frac{1}{\delta^2}$ such that

$$\#\mathfrak{L}_{2^{m_x}\delta, 2^{n_x}\lambda\delta}(x, j) \geq \frac{N}{(2\log_2 \frac{1}{\delta^2})^2}.$$

Indeed, if the inequality fails for every pair of such (m, n), summing over them would give us

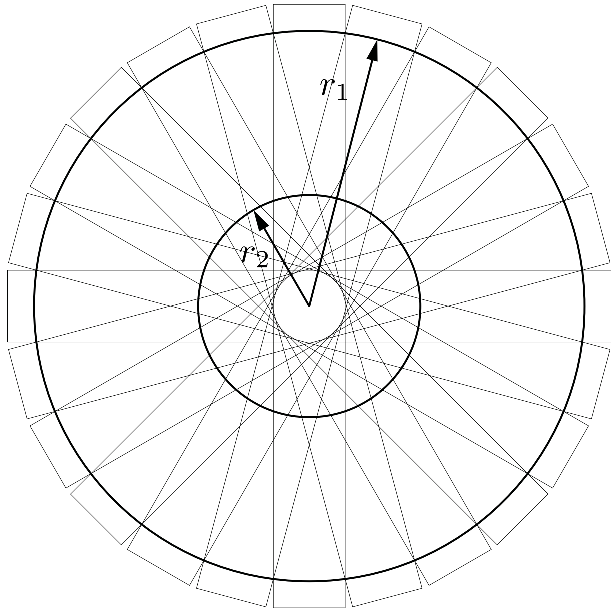

Figure 1.1: The overlapping of $\{l_k^\delta\}$.

a contradiction. Similarly, for a fixed j, using the pigeonhole principle again, we can find some uniform $1 \le m_j \le \log_2 \frac{1}{\delta}$ and $1 \le n_j \le \log_2 \frac{1}{\delta^2}$ such that (1.2.5) holds for all such fixed j. Finally, since there are $M/2$ values of j satisfying (1.2.6), if we use pigeonhole principle one more time, we can choose $\theta = 2^m\delta, \sigma = 2^n\lambda\delta$, so that (1.2.5) holds for at least $M/(2(\log_2 1/\delta^2))^2$ many values of j, finishing the proof. $\square$

Remark: The reason that we need σ to go down to the scale $\lambda\delta$ instead of δ is that we only have $\lambda|T_j^\delta|$ portion of each T_j^δ to apply pigeonhole principle, but this does not hurt us thanks to the fact that $\lambda \ge \delta$. Furthermore, noting that for such θ, σ that fulfill $\mathrm{II}_{\theta,\sigma}$, we must have

$$\lambda \lesssim_\epsilon \frac{\sigma}{\theta} \lesssim 1. \tag{1.2.7}$$

This will be crucial to extend the proof in [17] to dimension $d \ge 3$.

1.2.2 Auxiliary maximal function

First we prove a simple geometric lemma which will be useful in our proof and can be easily generalized to the constant curvature setting.

Lemma 2. *Let $0 < r_2 \le r_1 < 1$, and take a maximal δ-separated subset $\{v_k\}$ on $r_1 S^{d-2}$. Let l_k^δ be the δ-neighborhood of the line passing through the origin with direction v_k, then the number of* *overlaps of* $\{l_k^\delta\}$ *at some point* $y \in r_2 S^{d-2}$ *is at most*

$$C\left(\frac{r_1}{r_2}\right)^{d-2},$$

which implies

$$\sum_k \chi_{l_k^\delta \cap \{y' : |y'| \in [r_2/2, r_2)\}}(y') \lesssim \left(\frac{r_1}{r_2}\right)^{d-2}.$$

Proof. See Figure 1.1. We consider two cases. First, if $r_2 \leq \delta$, then the total number of overlaps is trivially bounded by the cardinality of the δ-separated subset, which is $C\left(\frac{r_1}{\delta}\right)^{d-2} \leq C\left(\frac{r_1}{r_2}\right)^{d-2}$.

On the other hand, if $r_2 > \delta$, then the points $\{r_2 v_k\}$ will be $\frac{r_2\delta}{r_1}$-separated on $r_2 S^{d-2}$, the number of overlaps of $\{l_k^\delta\}$ is bounded by

$$C\frac{\delta^{d-2}}{(\frac{r_2\delta}{r_1})^{d-2}} \sim C\left(\frac{r_1}{r_2}\right)^{d-2}.$$

. □

Remark: It is easy to extend this simple lemma to manifolds with constant curvature. One just need to notice that if there are two geodesic segements $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc length with $\gamma_1(0) = \gamma_2(0)$ and $\angle(\gamma_1, \gamma_2) = \beta$, then the distance $l(r)$ between $\gamma_1(r)$ and $\gamma_2(r)$ would satisfy

$$cr\beta \leq l(r) \leq Cr\beta,$$

where c, C only depend on the curvature, providing $r \leq \min\{1, \frac{1}{2}(\text{injectivity radius})\}$.

Within this section, we fix j and consider the tube $\boldsymbol{T}^\delta = T_{\xi^j}^\delta$. We may assume without loss of generality that the central axis γ_j of $\boldsymbol{T}^\delta$ is parallel to e_1, where $\{e_1, e_2, \ldots, e_d\}$ is an orthogonal normal basis of $\mathbb{R}^d$. For $y \in \mathbb{R}^d$, $\xi \in S^{d-1}$, we write $y = (y_1, y') = (y_1, y_2, y'')$, $\xi = (\xi_1, \xi') = (\xi_1, \xi_2, \xi'')$, where $y', \xi' \in \mathbb{R}^{d-1}, y'', \xi'' \in \mathbb{R}^{d-2}$ respectively.

We can now define the auxiliary maximal function as

$$A_\delta^{\theta,\sigma}(f)(\xi) = \sup_{T_\xi^\delta : \boldsymbol{T}^\delta \cap T_\xi^\delta \neq \emptyset, \angle(\boldsymbol{T}^\delta, T_\xi^\delta) \in [\theta/2, \theta)} \frac{1}{|T_\xi^\delta|} \int_{T_\xi^\delta \cap \{y : |y'| \in [\sigma/2, \sigma]\}} |f(y)|\, dy,$$

and define $A_\delta^{\theta,\sigma}(f)(\xi)$ to be zero if $\angle(e_1, \xi)$ is outside the interval $[\theta/2, \theta)$.

Theorem 3. *For the auxiliary maximal function* $A_\delta^{\theta,\sigma}$, *we have*

$$\|A_\delta^{\theta,\sigma}(f)\|_{L^2(S^{d-1})} \lesssim \left(\log\frac{1}{\delta}\right)^{\frac{1}{2}} \left(\frac{\theta}{\sigma}\right)^{\frac{d-2}{2}} \|f\|_{L^2(\mathbb{R}^d)}. \tag{1.2.8}$$

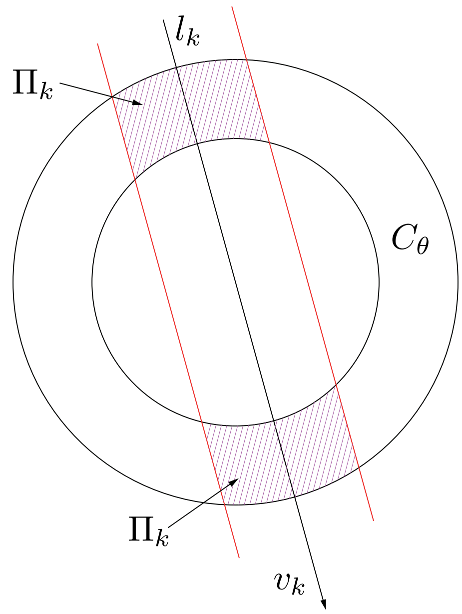

Figure 1.2: Π_k in $\mathbb{R}^{d-1}$.

Proof. For the sake of simplicity, we fix θ, σ and write $A_\delta^{\theta,\sigma}(f)$ simply as $A(f)$. Clearly, it suffices to estimate the integral

$$\int_{S_+^{d-1}} |A(f)|^2(\xi)\, dS,$$

where S_+^{d-1} is the upper half-sphere $\{\xi \in S^{d-1} : \xi_1 \geq 0\}$, and dS is the corresponding surface measure.

Since $\angle(\xi, e_1) \in [\theta/2, \theta)$, we see that $\sin\theta/2 \leq |\xi'| < \sin\theta$. Let

$$C_\theta = \{\xi' \in \mathbb{R}^{d-1} : \sin\theta/2 \leq |\xi'| < \sin\theta\}.$$

Take a maximal $\frac{\delta}{\sin\theta}$-separated subset $\{v_k\}$ of S^{d-2}, which has cardinality comparable to $(\frac{\theta}{\delta})^{d-2}$. Let l_k be the line passing through the origin with direction v_k, and l_k^δ denotes the δ-neighborhood of l_k. Let $\Pi_k = l_k^\delta \cap C_\theta$, and note that $\{\sin\theta \cdot v_k\}$ is a maximal δ-separated subset in $\sin\theta \cdot S^{d-2}$, so we must have $\cup_k \Pi_k \supset C_\theta$. Again by the maximality of $\{v_k\}$, we see that $\{\Pi_k\}$ has bounded overlap, so they are essentially disjoint pairwise. Indeed, we can take a new collection of sets $\{\Gamma_k\}$ which also covers C_θ, with $\Gamma_1 = \Pi_1$, and $\Gamma_k = \Pi_k \setminus \cup_{j=1}^{k-1} \Pi_j$. Clearly each Γ_k will be nonempty and they are pairwise disjoint.

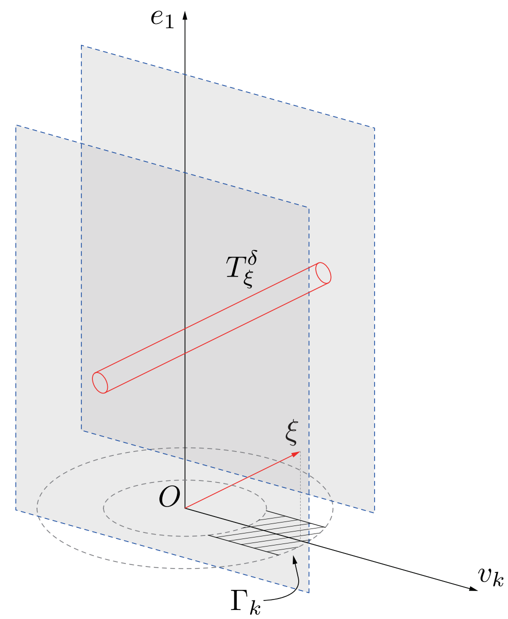

Figure 1.3: T_ξ^δ contained in V_k

Taking $r_1 = \theta \sim \sin\theta, r_2 = \sigma$ in Lemma 1, we see that

$$\sum_k \chi_{l_k^\delta \cap \{y' : |y'| \in [\sigma/2, \sigma)\}}(y') \lesssim \left(\frac{\theta}{\sigma}\right)^{d-2}.$$

Consider $\xi' \in \Gamma_k$ for some k. Remember that we require $\boldsymbol{T}^\delta \cap T_\xi^\delta \neq \emptyset$, so the tube T_ξ^δ with direction $\xi = (\sqrt{1 - |\xi'|^2}, \xi')$ must lie in a 10δ-neighborhood $H_k^{10\delta}$ of the 2-plane

$$H_k = \operatorname{span}\{e_1, (0, v_k)\},$$

see Figure 2.2. Let

$$V_k = \left\{y \in \mathbb{R}^d : |y_1| \le 1\right\} \cap H_k^{10\delta},$$

then clearly

$$\sum_k \chi_{V_k \cap \{y : |y'| \in [\sigma/2, \sigma)\}}(y) \lesssim \sum_k \chi_{l_k^\delta \cap \{y' : |y'| \in [\sigma/2, \sigma)\}}(y') \lesssim \left(\frac{\theta}{\sigma}\right)^{d-2}.$$

Now we begin to estimate $\int_{S_+^{d-1}} |A(f)|^2(\xi)\, dS$. We claim that it suffices to prove the following L^2 estimate for each V_k,

$$\|A(f\chi_{V_k})\|_{L^2(\{\xi \in S_+^{d-1} : \xi' \in \Gamma_k\})} \lesssim \left(\log\frac{1}{\delta}\right)^{\frac{1}{2}} \|f\chi_{V_k}\|_{L^2}. \tag{1.2.9}$$

Indeed, noting that $\theta \le 1$,

$$\begin{aligned}\int_{S_+^{d-1}} |A(f)|^2(\xi)\, dS &\lesssim \int_{\mathbb{R}^{d-1}} |A(f)|^2(\sqrt{1-|\xi'|^2}, \xi')\, d\xi' \\ &\lesssim \sum_k \int_{\Gamma_k} |A(f\chi_{V_k})|^2(\sqrt{1-|\xi'|^2}, \xi')\, d\xi' \\ &\lesssim \left(\log \frac{1}{\delta}\right) \sum_k \int_{\mathbb{R}^d} |(f\chi_{V_k})|^2\, dy \\ &\lesssim \left(\log \frac{1}{\delta}\right) \left(\frac{\theta}{\sigma}\right)^{d-2} \|f\|_2^2.\end{aligned}$$

It remains to prove (1.2.9). Without loss of generality, we assume $(0, v_k) = e_2$, and only consider functions f with support inside $V_k \cap \{y : |y'| \in [\sigma/2, \sigma)\}$.

For $|y''| < 10\delta$, we let $\mathbf{P}(y'')$ denote the 2-plane that passes through the point $(0, 0, y'')$ and is parallel to span $\{e_1, e_2\} = H_k$. Then for any $\xi' \in \Gamma_k$, $\xi = (\xi_1, \xi')$, the set $\mathbf{P}(y'') \cap T_\xi^\delta$ is the intersection of a 2-plane with a d-dimensional δ-tube, so it can always be contained in some 2-dimensional tube $t^\delta(y'')$ with direction $\frac{(\xi_1,\xi_2)}{\sqrt{\xi_1^2+\xi_2^2}}$.

Take $r = \sqrt{1-|\xi''|^2}$ then $r \sim \sqrt{1 - C\delta^2} \ge \frac{1}{2}$, and let M_δ be the standard 2-dimensional Kakeya maximal function, then we have

$$\begin{aligned}\delta^{-(d-1)} \int_{T_\xi^\delta} |f(y)|\, dy &= \delta^{-(d-1)} \int_{|y''| \le 10\delta} dy'' \int_{\mathbf{P}(y'') \cap T_\xi^\delta} |f(y_1, y_2, y'')|\, dy_1 dy_2 \\ &\le \delta^{-(d-1)} \int_{|y''| \le 10\delta} dy'' \int_{t^\delta(y'')} |f(y_1, y_2, y'')|\, dy_1 dy_2 \\ &\lesssim \delta^{-(d-2)} \int_{|y''| \le 10\delta} M_\delta(f(\,\cdot\,, y'')) \left(\frac{\sqrt{r^2 - |\xi_2|^2}, \xi_2}{r}\right) dy'',\end{aligned}$$

therefore,

$$A(f)(\xi) \lesssim \delta^{-(d-2)} \int_{|y''| \le 10\delta} M_\delta(f(\,\cdot\,, y'')) \left(\frac{\sqrt{r^2 - |\xi_2|^2}, \xi_2}{r}\right) dy''.$$

Noticing that if ϕ is some proper parameter for the subset of S^1 where $|\xi_2| \le \sin\theta \le \sin 1$, then $|\frac{d\phi}{d\xi_2}|$

is bounded below by some constant. Minkowski's inequality gives us

$$\begin{aligned}
&\left(\int_{|\xi_2|\le\sin\theta}|A(f)(\xi')|^2 d\xi_2\right)^{\frac12}\\
&\lesssim \delta^{-(d-2)}\int_{|y''|\le 10\delta} dy''\left(\int_{|\xi_2|\le\sin\theta}|M_\delta(f(\,\cdot\,,y''))|^2\left(\frac{\sqrt{r^2-|\xi_2|^2},\xi_2}{r}\right)d\xi_2\right)^{\frac12}\\
&\lesssim \delta^{-(d-2)}\int_{|y''|\le 10\delta} dy''\left(\int_{S^1}|M_\delta(f(\,\cdot\,,y''))|^2(\phi)\,d\phi\right)^{\frac12}\\
&\lesssim \left(\log\frac1\delta\right)^{\frac12}\delta^{-(d-2)}\int_{|y''|\le 10\delta}\|f(\,\cdot\,,y'')\|_{L^2(y_1,y_2)}\,dy''\\
&\lesssim \left(\log\frac1\delta\right)^{\frac12}\delta^{-\frac{(d-2)}{2}}\|f\|_{L^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\int_{\{\xi\in S^{d-1}_+:\xi'\in\Gamma_k\}}|A(f)(\xi)|^2\,dS\right)^{\frac12} &\lesssim \left(\int_{\Gamma_k}|A(f)|^2(\sqrt{1-|\xi'|^2},\xi')\,d\xi'\right)^{\frac12}\\
&\le \left(\int_{\{\xi':|\xi_2|\le\sin\theta,|\xi''|\le 10\delta\}}|A(f)|^2(\sqrt{1-|\xi'|^2},\xi')\,d\xi'\right)^{\frac12}\\
&= \left(\int_{|\xi''|\le 10\delta}d\xi''\int_{|\xi_2|\le\sin\theta}|A(f)(\xi')|^2\,d\xi_2\right)^{\frac12}\\
&\lesssim \left(\log\frac1\delta\right)^{\frac12}\delta^{-\frac{(d-2)}{2}}\left(\int_{|\xi''|\le 10\delta}\|f\|^2_{L^2}\,d\xi''\right)^{\frac12}\\
&\lesssim \left(\log\frac1\delta\right)^{\frac12}\|f\|_{L^2}.
\end{aligned}$$

□

Remark: The key difference between our auxiliary maximal function estimate and that in [17] is that we reduce to the optimal 2-dimensional L^2 Kakeya bound for 2-planes rather than reducing to $(d-1)$-dimensional case for hyperplanes. In this way, instead of a $\delta^{-(d-3)/2}$ loss, the extra factor $(\theta/\sigma)^{(d-2)/2}$ we have can be handled using (1.2.7). This is in fact natural if one looks back to Wolff's original hairbrush argument, the 2-dimensional L^2 estimate for 2-planes is enough to justify that the "bristles" are essentially separated. In other words, reducing to 2-dimensional case already gives the best possible result for the hairbrush argument, so we do not expect improvements by reducing to the $(d-1)$-dimensional case.

1.2.3 A key lemma

From now on, let N be the number that fulfills both case I and $\mathrm{II}_{\theta,\sigma}$ as in Lemma 1, and again we fix an index j such that $\boldsymbol{T}^\delta = T^\delta_{\xi^j}$ satisfies $\mathrm{II}_{\theta,\sigma}$. Using our L^2 estimate for the auxiliary maximal function from last section, we will show that we can generalize Proposition 2.5 in [22] and Lemma 5.2 in [17] to any dimension $d \geq 3$, which was the part where Wolff used induction on scales in his paper.

Lemma 3. *For any $\epsilon > 0$, any point a*

$$|E \cap B(a, \delta^\epsilon \lambda)^c \cap \boldsymbol{T}^\sigma| \gtrsim_\epsilon \lambda^d N \sigma \delta^{d-2}. \tag{1.2.10}$$

Proof. We claim that it suffices to show

$$|E \cap \boldsymbol{T}^\sigma| \gtrsim_\epsilon \lambda^d N \sigma \delta^{d-2}. \tag{1.2.11}$$

Indeed, noticing the fact that for δ sufficiently small, the set $E \cap B(a, \delta^\epsilon \lambda)^c \cap \boldsymbol{T}^\sigma$ has size at least $\frac{1}{2}$ of the size of $E \cap \boldsymbol{T}^\sigma$, we can replace E by $E \cap B(a, \delta^\epsilon \lambda)^c$ in (1.2.11) and get (1.2.10). See [22] and Proposition 5.2 of [17] for details.

For the tube $\boldsymbol{T}^\delta$, we denote

$$\boldsymbol{S}^\delta = \boldsymbol{T}^\delta \cap E \cap \left\{ x : \#\mathfrak{L}_{\theta,\sigma}(x,j) \geq 2^{-2} N \left(\log_2 \frac{1}{\delta^2} \right)^{-2} \right\}.$$

By the definition of $\mathfrak{L}_{\theta,\sigma}(x,j)$, we see that there is a $M_0 \in (0, M]$ and a subcollection $\{T^\delta_{i_k}\}_x$ of $\{T^\delta_i\}_{i=0}^M$ that are in $\mathfrak{L}_{\theta,\sigma}(x,j)$ for each x, so that if we let x run through every point in $\boldsymbol{S}^\delta$, and take the union of these subcollections to get $\{T^\delta_{i_k}\}_{k=1}^{M_0}$, then we will have

$$\sum_{k=1}^{M_0} \chi_{T^\delta_{i_k}} \geq \frac{N}{2^2} \left(\log_2 \frac{1}{\delta^2} \right)^{-2} \text{ on } \boldsymbol{S}^\delta.$$

Recall that two δ-tubes which intersect at angle θ would have intersection measure less than $C\frac{\delta^d}{\theta}$, so we have

$$|\boldsymbol{S}^\delta| \lesssim_\epsilon N^{-1} \int_{\boldsymbol{T}^\delta} \sum_{k=1}^{M_0} \chi_{T^\delta_{i_k}}(x)dx \leq N^{-1} \sum_{k=1}^{M_0} |T^\delta_{i_k} \cap \boldsymbol{T}^\delta| \lesssim \frac{M_0 \delta^d}{N\theta},$$

together with the simple fact

$$|\boldsymbol{S}^{\delta}| \gtrsim_{\epsilon} \lambda |\boldsymbol{T}^{\delta}|,$$

we conclude

$$M_0 \gtrsim_{\epsilon} \theta \delta^{-1} N \lambda. \tag{1.2.12}$$

Now, consider the average of the function $f = \chi_{E \cap \{y : \mathrm{dist}(y, \gamma_j) \in [\sigma/2, \sigma)\}}$ over $T^{\delta}_{i_k}$, we have

$$\delta^{-(d-1)} \int_{T^{\delta}_{i_k}} f(y)\, dy = \delta^{-(d-1)} \left| T^{\delta}_{i_k} \cap E \cap \{y : \mathrm{dist}(y, \gamma_j) \in [\sigma/2, \sigma)\} \right| \gtrsim_{\epsilon} \lambda.$$

On the other hand,

$$\delta^{-(d-1)} \int_{T^{\delta}_{i_k}} f(y)\, dy \le A^{\theta,\sigma}_{\delta}(f)(\xi_{i_k}).$$

After combining these two inequalities, we square both sides, multiply both sides byδ^{d-1} and sum up with respect to $k = 1, \ldots, M_0$, then we have

$$\begin{aligned} M_0 \delta^{d-1} \lambda^2 &\lesssim_{\epsilon} \sum_{k=1}^{M_0} |A^{\theta,\sigma}_{\delta}(f)(\xi_{i_k})|^2 \delta^{d-1} \\ &\lesssim \|A^{\theta,\sigma}_{\delta}(f)(\xi)\|^2_{L^2(S^{d-1})} \\ &\lesssim_{\epsilon} \frac{\theta^{d-2}}{\sigma^{d-2}} |E \cap \{y : \mathrm{dist}(y, \gamma_j) \in [\sigma/2, \sigma)\}| \\ &\lesssim_{\epsilon} \frac{\theta}{\sigma \lambda^{d-3}} |E \cap \boldsymbol{T}^{\sigma}|, \end{aligned}$$

where we used the maximality of the $\{\xi_k\}$, (1.2.8) and (1.2.7). Using (1.2.12) for the estimate of M_0, we get (1.2.11). ☐

1.2.4 Completion of the proof

We shall give the estimates corresponding to high and low multiplicity cases separately, and we start with the simpler one.

Lemma 4. *For N satisfy* I,

$$|E| \gtrsim \frac{\lambda M \delta^{d-1}}{N}. \tag{1.2.13}$$

Proof. Let $E_0 = \{x \in E : \sum_{k=1}^{M} \chi_{T^{\delta}_k(x)} \le N\}$. Recalling that N fulfills case I, we know $|T^{\delta}_i \cap E_0| \ge$

$\lambda|T_i^\delta|/2$ for at least $M/2$ values of $i = i_k$. Thus

$$|E| \geq \left|\bigcup_{k=1}^{M/2}(E_0 \cap T_{i_k}^\delta)\right| \geq N^{-1}\sum_{k=1}^{M/2}|E_0 \cap T_{i_k}^\delta| \gtrsim \frac{\lambda M\delta^{d-1}}{N}.$$

In order to estimate the high multiplicity case, we need to establish a bush argument for the collection of hairbrushes $\{E \cap T_j^\sigma\}$, where the following lemma plays a key role.

Lemma 5. *Suppose there are M tubes $\{T_j^\sigma\}_{j=1}^M$ such that $j \neq j'$ and $T_j^\sigma \cap T_{j'}^\sigma \neq \emptyset$ implies $\angle(T_j^\sigma, T_{j'}^\sigma) \geq \gamma$ for some $0 < \gamma < \frac{\pi}{2}$. Assume also that for some $\rho > 0$ and any $a \in \mathbb{R}^d$, there are M_0 such tubes satisfying*

$$\rho|T_j^\sigma| \leq |T_j^\sigma \cap E \cap B(a, \sigma/\gamma)^c|. \tag{1.2.14}$$

Then we have

$$|E| \geq \frac{\rho\sigma^{d-1}M_0^{1/2}}{2}. \tag{1.2.15}$$

Proof. By relabeling the indices, we have a sequence $\{T_j^\sigma\}_{j=1}^{M_0}$ satisfying

$$\rho\sigma^{d-1}M_0 \leq \int_E \sum_{j=1}^{M_0}\chi_{T_j^\sigma}(x)dx.$$

Thus, there exists an $x_0 \in E$ such that

$$\sum_{j=1}^{M_0}\chi_{T_j^\sigma}(x_0) \geq \frac{\rho\sigma^{d-1}M_0}{2|E|}.$$

Noting that the diameter of $T_{j'}^\sigma \cap T_j^\sigma$ is at most σ/γ, so $B(x_0, \sigma/\gamma)^c \cap T_j^\sigma \cap T_{j'}^\sigma = \emptyset$, we have

$$|E| \geq \left|E \cap B(x_0, \sigma/\gamma)^c \cap \bigcup_{\{j: x_0 \in T_j^\sigma\}} T_j^\sigma\right| \geq \sum_{\{j: x_0 \in T_j^\sigma\}} |E \cap B(x_0, \sigma/\gamma)^c \cap T_j^\sigma| \geq \frac{\rho^2\sigma^{2(d-1)}M_0}{4|E|}.$$

□

Lemma 6. *Let N satisfy $\mathrm{II}_{\theta,\sigma}$, then we have*

$$|E| \gtrsim_\epsilon \lambda^{d+1}N(M\delta^{d-1})^{\frac{1}{d-1}}\delta^{d-2} \tag{1.2.16}$$

Proof. By the multiplicity argument, we know that for some suitable constant c, there are at least

$$[cM(\log_2 \frac{1}{\delta^2})^{-2}]$$

many tubes in $\mathrm{II}_{\theta,\sigma}$, denote them by

$$\{T_j^\delta\}_{j=1}^{[cM(\log_2 \frac{1}{\delta^2})^{-2}]}.$$

Let

$$\gamma = \frac{\sigma}{\delta^\epsilon \lambda},$$

then clearly $\gamma \geq \delta^{1-\epsilon}$. If $\gamma \geq \frac{\pi}{2}$, then (1.2.16) follows directly from (1.2.10). Otherwise, take a maximal γ-separated subset of $\{\xi_j\}_{j=1}^{[cM(\log_2 \frac{1}{\delta^2})^{-2}]}$ and denote the size of this subset to be M_0. By maximality, we see easily

$$M_0 \gtrsim \frac{M}{\left(\log_2 \frac{1}{\delta^2}\right)^2} \delta^{d-1} \left(\frac{\delta^\epsilon \lambda}{\sigma}\right)^{d-1} \gtrsim_\epsilon M\delta^{d-1} \left(\frac{\lambda}{\sigma}\right)^{d-1},$$

and using (1.2.10) one may easily check that if we let $\rho = C_\epsilon \lambda^d \sigma^{2-d} \delta^{d-2+\epsilon} N$ for some proper constant C_ϵ then all requirements of Lemma 5 are fulfilled, so we have

$$\begin{aligned}
|E| &\gtrsim_\epsilon \lambda^d \sigma^{2-d} \delta^{d-2} N \cdot \sigma^{d-1} M_0^{\frac{1}{2}} \\
&\geq \lambda^d \sigma \delta^{d-2} N M_0^{\frac{1}{d-1}} \\
&\gtrsim_\epsilon \lambda^d \sigma \delta^{d-2} N \left(M\delta^{d-1} \left(\frac{\lambda}{\sigma}\right)^{d-1} \right)^{\frac{1}{d-1}} \\
&= \lambda^d \sigma \delta^{d-2} N (M\delta^{d-1})^{\frac{1}{d-1}} \lambda \sigma^{-1} \\
&\geq \lambda^{d+1} N (M\delta^{d-1})^{\frac{1}{d-1}} \delta^{d-2},
\end{aligned}$$

where we used the fact that $M_0^{1/2} \geq M_0^{1/(d-1)}$ since $M_0 \geq 1$ and $d \geq 3$. $\square$

Now if we take the geometric mean of (1.2.16) and (1.2.13), we get (1.2.3), completing the proof.

1.3 Nikodym-type maximal function in spaces of constant curvature

Once we know how to prove Wolff's result without appealing to induction on scales, it is easy to generalize Sogge's result for Nikodym maximal function in 3-dimensional spaces of constant curvature to any dimension $d \geq 3$. This section is somewhat parallel to the last section. Throughout this section, we fix a dimension $d \geq 3$ and use C, c to denote various constants that only depend on the curvature of the manifold.

Let (M^d, g) be a Riemannian manifold. Throughout this section, we fix a number $\alpha > 0$ that is smaller than $\min\{1, \frac{1}{2}\mathrm{inj}\, M^d\}$, where $\mathrm{inj}\, M^d$ denotes the injectivity radius of M^d. Let γ_x denote any geodesic passing through $x \in M^d$ of length α. Using the metric, we let

$$T_x^\delta = \{y \in M^d : \mathrm{dist}(y, \gamma_x) \leq \delta\}$$

be a tubular $\delta-$neighborhood around γ_x. We shall also sometimes use the notation $T_{\gamma_x}^\delta$ to denote the same tube. Now given a function f on M^d, we can define the Nikodym maximal function

$$f_\delta^{**}(x) = \sup \frac{1}{|T_x^\delta|} \int_{T_x^\delta} |f(y)|\, dy.$$

Since the Nikodym problem is local, Wolff's result (Theorem 1) implies if M^d has constant curvature 0, then we have

$$\|f_\delta^{**}\|_{L^q(M^d)} \lesssim_\epsilon \delta^{1-\frac{d}{p}} \|f\|_{L^p(M^d)},\ p = \frac{d+2}{2},\ q = (d-1)p'.$$

On the other hand, Sogge [22] showed that same bounds hold in the constant curvature case if $d = 3$ (Theorem 2).

In this section we prove Theorem 2, which extend Sogge's result to any dimension $d \geq 3$.

Clearly, the $L^1 \to L^\infty$ bounds are trivial, so it suffices to prove the following restricted weak-type estimate

$$|\{x \in M^d : (\chi_E)_\delta^{**}(x) \geq \lambda\}| \lesssim_\epsilon (\lambda^{-\frac{d+2}{2}} \delta^{\frac{2-d}{2}} |E|)^{\frac{2d-2}{d}}, \tag{1.3.1}$$

where E is a set contained in our coordinate patch.

Before turning to the proof of (1.3.1), we quote a useful geometric lemma which is in [18].

Lemma 7. *Suppose γ_1, γ_2 are geodesics of length α and assume that the γ_j belong to a fixed compact subset K of M^d. Suppose also $a \in T^\delta_{\gamma_1} \cap T^\delta_{\gamma_2}$. Then there is a constant $c > 0$, depending on (M^d, g) and K, so if*

$$\angle(T^\delta_{\gamma_1}, T^\delta_{\gamma_2}) \geq \frac{\delta}{c\lambda},$$

then we have

$$(T^\delta_{\gamma_1} \cap T^\delta_{\gamma_2}) \setminus B(a, \lambda) = \emptyset.$$

Here we are using the induced metric on the unit tangent bundle to define the angle between two geodesics (tubes) γ_1, γ_2 of length α

$$\angle(T^\delta_{\gamma_1}, T^\delta_{\gamma_2}) = \angle(\gamma_1, \gamma_2) = \min_{x_j \in \gamma_j, \tau_j = \gamma_j'|_{\gamma_j = x_j}} \mathrm{dist}_{UTM^d}((x_1, \tau_1), (x_2, \tau_2)).$$

Here $\gamma_j'|_{\gamma_j = x_j}$ denotes a unit tangent vector at x_j.

As in [22], [28] and [5], it is convenient to work with a discrete form of the problem.

We fix a geodesic γ_0 and work in Fermi normal coordinates near γ_0. To obtain these Fermi normal coordinates, we first fix a point $x_0 \in \gamma_0$ and then choose an orthonormal basis $\{e_k\}_{k=1}^d \subset T_{x_0} M^d$ with e_1 being a unit tangent vector of γ_0 at x_0. Using parallel transport, one propagates this basis to every point of γ_0. If we choose $\gamma_0(s)$ to be the arc length parameterization of γ_0 with $\gamma_0(0) = x_0$ and $\gamma_0'(0) = e_1$, then the resulting vectors $\{e_k(s)\}$ will be orthonormal in $T_{\gamma_0(s)} M^d$ and $\gamma'(s) = e_1(s)$. We then assign Fermi coordinates $(x_1, x_2, \ldots, x_d) = (x, x')$ to a point x, if it is the endpoint of the geodesic of length $|x'|$ starting at $\gamma_0(x_1)$ with tangent vector $(0, x')$.

These coordinates provide us with some good properties. First, the rays $t \to (x_1, tx')$ are geodesics orthogonal to $\boldsymbol{\gamma}$. Second, by construction we see that the vector fields ∂x_k are parallel along $\boldsymbol{\gamma}$. Also, these Fermi normal coordinates are unique up to rotations preserving the x_1-axis. See details in [22].

Now we fix a small number $c > 0$, and consider only the geodesics γ that, belong to the collection

$$G = \{\gamma_{x'} : (0, x') \in \gamma_{x'} \text{ for some } x', \angle(\gamma_{x'}, \gamma_0) \leq c\}.$$

Then for a large fixed constant C_0, we consider a $C_0\delta$-separated collection $\{x_j'\}_{j=1}^M$ of the set $\{(0, x') \in M^d : (\chi_E)^{**}_\delta(0, x') \geq \lambda\}$. For each j, we choose a tube T_j^δ to be the δ-tube about some $\gamma_{x_j'} \in G$ such that

$$|E \cap T_j^\delta| \geq \lambda |T_j^\delta|,$$

then (1.3.1) would follow from the uniform bounds

$$M\delta^{d-1} \lesssim_\epsilon (\lambda^{-\frac{d+2}{2}}\delta^{\frac{2-d}{2}}|E|)^{\frac{2d-2}{d}}, \tag{1.3.2}$$

Indeed, this inequality implies the slightly stronger version of (1.3.1), where the left hand side is replaced by $|\{(0,x') \in M^d : (\chi_E)^{**}_\delta(0,x') \geq \lambda\}|$, and we replace the maximal operator by one involving averaging over δ-tubes with central geodesics in G.

Note since the basepoints $\{x'_j\}$ of the tubes are δ-separated, we must have

$$\angle(T_j^\delta, T_i^\delta) > c\delta, \quad \text{if } i \neq j,$$

for some constant c. Now we use the exact same multiplicity argument which we used for the Kakeya problem in $\mathbb{R}^d$.

1.3.1 Multiplicity argument

Consider parameters $\theta \in [\delta, 1]$, and $\sigma \in [\lambda\delta, 1]$. First, for $1 \leq j \leq M$ and $x \in T_j^\delta$ fixed, let

$$\mathfrak{L}_\theta(x,j) = \{i : x \in T_i^\delta, \angle(T_j^\delta, T_i^\delta) \in [\theta/2, \theta)\}$$

index the tubes T_i^δ containing x which intersect the fixed tube T_j^δ at angle comparable to θ. Next, let

$$\mathfrak{L}_\sigma(x,j) = \{i : x \in T_i^\delta, |T_i^\delta \cap \{y \in E : \mathrm{dist}(y,\gamma_j) \in [\sigma/2, \sigma)\}| \geq (2\log_2 \frac{1}{\delta^2})^{-1}\lambda|T_i^\delta|\}$$

index the tubes T_i^δ containing x which intersect the fixed tube T_j^δ at x such that there is non-trivial portion of $T_i^\delta \cap E$ with distance to γ_j comparable to σ. Now let

$$\mathfrak{L}_{\theta,\sigma}(x,j) = \mathfrak{L}_\theta(x,j) \cap \mathfrak{L}_\sigma(x,j).$$

Then we have the following

Lemma 8. *There are $N \in \mathbb{N}$ and $\theta \in [\delta, 1]$, $\sigma \in [\lambda\delta, 1]$ that fulfills the following two cases*

I. (*Low multiplicity case*) *There are at least $M/2$ values of j for which*

$$\left|\left\{x \in T_j^\delta \cap E : \#\{i : x \in T_i^\delta\} \leq N\right\}\right| \geq \frac{\lambda}{2}|T_j^\delta|.$$

$\mathrm{II}_{\theta,\sigma}$. *(High multiplicity case at angle θ and distance σ) There are at least $M/(2(\log_2 1/\delta^2))^2$ many values of j for which*

$$\left|\left\{x \in T_j^\delta \cap E : \#\mathfrak{L}_{\theta,\sigma} \geq \frac{N}{(2\log_2 \frac{1}{\delta^2})^2}\right\}\right| \geq \frac{\lambda}{(4\log_2 \frac{1}{\delta^2})^2}|T_j^\delta|. \tag{1.3.3}$$

The proof is identical to that of Lemma 1. We also have the same bound for σ/θ as in the remark of Lemma 1 for the same reason.

$$\lambda \lesssim_\epsilon \frac{\sigma}{\theta} \lesssim 1. \tag{1.3.4}$$

1.3.2 Auxiliary maximal function

Throughout this section, we fix a tube $\boldsymbol{T}^\delta$. We follow Sogge's strategy in [22] closely and generalize it to any dimension $d \geq 3$. We work in the Fermi normal coordinates near the central geodesic $\boldsymbol{\gamma}$ of $\boldsymbol{T}^\delta$.

We now define the auxiliary maximal function for

$$A_\delta^{\theta,\sigma}(f)(x') = \sup_{T_{\gamma_{x'}} \in S_{x'}} \frac{1}{|T^\delta_{\gamma_{x'}}|} \int_{T^\delta_{\gamma_{x'}} \cap \{y : |y'| \in [\sigma/2, \sigma]\}} |f(y)|\, dy,$$

where the supremum runs through the collection of tubes

$$S_{x'} = \{T^\delta_{\gamma_{x'}} : (0, x') \in \gamma_{x'},\, \gamma_{x'} \cap \boldsymbol{\gamma} \neq \emptyset,\, \angle(\gamma_{x'}, \boldsymbol{\gamma}) \in [\theta/2, \theta)\},$$

and define $A_\delta^{\theta,\sigma}(f)(x')$ to be zero if $S_{x'} = \emptyset$.

Theorem 4. *With $A_\delta^{\theta,\sigma}$ as above, we have*

$$\|A_\delta^{\theta,\sigma}(f)\|_{L^2} \lesssim \left(\log \frac{1}{\delta}\right)^{\frac{1}{2}} \left(\frac{\theta}{\sigma}\right)^{\frac{d-2}{2}} \|f\|_{L^2}. \tag{1.3.5}$$

Proof. Write $A_\delta^{\theta,\sigma}(f)$ simply as $A(f)$. The proof is very similar to the proof of Theorem 1. We estimate the integral

$$\int |A(f)|^2(x')dx.$$

Noticing that if we require $S_{x'} \neq \emptyset$, then $|x'| \leq C \sin\theta$ for some C that only depends on the curvature. We define the subset C_θ in the base hyperplane $\{x \in (M^d, g) : x_1 = 0\}$ by

$$C_\theta = \{x' : |x'| \leq C \sin\theta\}.$$

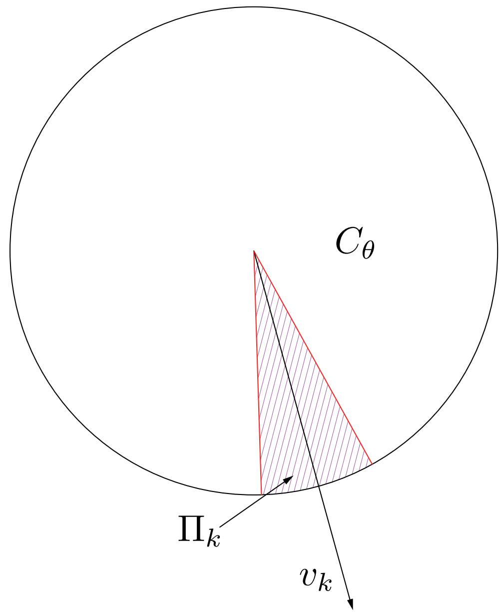

Figure 1.4: Π_k in the base hyperplane

Take a maximal $\frac{\delta}{\sin\theta}$-separated subset $\{v_k\}$ of S^{d-2}, which has cardinality comparable to $(\frac{\theta}{\delta})^{d-2}$. Let $\Pi_k \subset C_\theta$ be the conic set in $\{x : x_1 = 0\}$ such that

$$\Pi_k \cap \sin\theta \cdot S^{d-1} = B(\sin\theta \cdot v_k, \delta) \cap \sin\theta \cdot S^{d-1},$$

see Figure 2.4. As in proof of Theorem 1, we must have $\cup_k \Pi_k \supset C_\theta$. And by the maximality of $\{v_k\}$, we can further assume Π_k's to be pairwise disjoint.

Consider $x' \in \Gamma_k$ for some k. Let

$$H_k = \operatorname{span}\{e_1, (0, v_k)\},$$

Then H_k would be totally geodesic as a Fermi 2-plane. Remember that we require $\boldsymbol{\gamma} \cap \gamma_{x'} \neq \emptyset$, so any tube $T^\delta_{\gamma_{x'}} \in S_{x'}$ must lie in a $C\delta$-neighborhood $H_k^{C\delta}$ for some k. Where C is again some suitable constant that only depends on the curvature. Let

$$V_k = \{x : |x_1| \le 1\} \cap H_k^{C\delta},$$

then by the remark of Lemma 2, we have

$$\sum_k \chi_{V_k \cap \{y : |y'| \in [\sigma/2, \sigma)\}}(y) \lesssim \left(\frac{\theta}{\sigma}\right)^{d-2}.$$

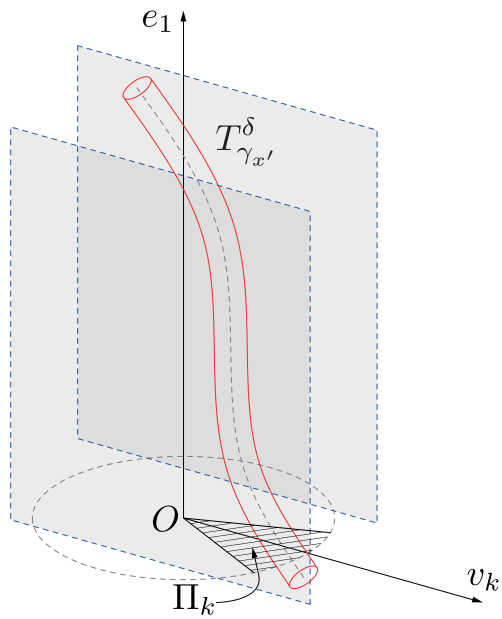

Figure 1.5: $T^{\delta}_{\gamma_{x'}}$ contained in V_k

Similar to the Kakeya case in $\mathbb{R}^d$, we conclude using the above fact and a twofold application of Schwarz's inequality, the theorem would follow from the following L^2 estimate for each k,

$$\|A(f\chi_{V_k})\|_{L^2(\Pi_k)} \lesssim \left(\log\frac{1}{\delta}\right)^{\frac{1}{2}} \|f\chi_{V_k}\|_{L^2}. \tag{1.3.6}$$

To prove (1.3.6), we need a curved version of the 2-dimensional Nikodym maximal inequality.

To state it we now suppose that (M^2, g) is a 2-dimensional Riemannian manifold. If we fix a geodesic $\gamma_0 \subset M^2$ of length $\alpha \le \min\{1, (\mathrm{inj}M^2)/2\}$, we consider all geodesic γ of this length which are close to γ_0. Let $\gamma_1(t)$ be a geodesic which intersects γ_0 orthogonally and is parameterized by arc length. We set

$$M_\delta g(t) = \sup_{\gamma_1(t)\in\gamma} \delta^{-1} \int_{\{y:\mathrm{dist}(y,\gamma)\le\delta\}} |g(y)|\,dy. \tag{1.3.7}$$

We claim (1.3.6) would follow from

$$\|M_\delta g\|_{L^2(dt)} \lesssim \left(\log\frac{1}{\delta}\right)^{\frac{1}{2}} \|g\|_{L^2(M^2)}. \tag{1.3.8}$$

This is (2.43) in [22], and we refer readers to [22] and [18] for the proof.

Now we show how (1.3.8) implies (1.3.6). We use the same trick as we did for the Kakeya problem in Euclidean case. Without loss of generality, we fix k, assume $e_2 = (0, v_k)$ and only consider functions f with support contained in $V_k \cap \{y : |y'| \in [\sigma/2, \sigma)\}$.

Let $\mathbf{P}(s)$ be the surface that corresponds to the 2-plane $\{y \in (M^d, g) : y = (y_1, y_2, s)\}$ with volume element dV_s, where s is a $(d-2)$-dimensional parameter for the collection of those 2-planes with $|s| \le C\delta$. Since $\mathbf{P}(0) = \operatorname{span}\{e_1, e_2\}$ is a totally geodesic 2-plane and we are in constant curvature case, $|dV_0| \sim |dy_1 dy_2|$.

For any $x = (0, x') = (0, x_2, x'') \in \Pi_k$, we consider the integral over the cross section $\mathbf{P}(s) \cap T^{\delta}_{\gamma_{x'}}$. Clearly, the projection of this cross section onto $\mathbf{P}(0)$ is contained in $\mathbf{P}(0) \cap T^{C'\delta}_{\gamma_{x'}}$ for some constant C'. Noticing the fact that dV_s varies smoothly with respect to s, we see that for fixed s with $|s| \le C\delta$

$$\int_{\mathbf{P}(s)\cap T^{\delta}_{\gamma_{x'}}} |f(y_1, y_2, s)|\, dV_s \lesssim \int_{\mathbf{P}(0)\cap T^{C'\delta}_{\gamma_{x'}}} |f(y_1, y_2, s)|\, dV_0 \lesssim \int_{\mathbf{P}(0)\cap T^{C'\delta}_{\gamma_{x'}}} |f(y_1, y_2, s)|\, dy_1 dy_2.$$

Since $\mathbf{P}(0)$ is totally geodesic, $\mathbf{P}(0) \cap T^{C'\delta}_{\gamma_{x'}}$ is contained in $\mathbf{P}(0) \cap T^{C''\delta}_{\gamma_{(0,x_2)}}$ for some $\gamma_{(0,x_2)}$ and C''. Then we have

$$\begin{aligned}
\delta^{-(d-1)} \int_{T^{\delta}_{\gamma_{x'}}} |f(y)|\, dy &= \delta^{-(d-1)} \int_{|y''|\le C\delta} dy'' \int_{\mathbf{P}(y'')\cap T^{\delta}_{\gamma_{x'}}} |f(y_1, y_2, y'')|\, dV_{y''} \\
&\le \delta^{-(d-1)} \int_{|y''|\le C\delta} dy'' \int_{\mathbf{P}(0)\cap T^{C'\delta}_{\gamma_{x'}}} |f(y_1, y_2, y'')|\, dy_1 dy_2 \\
&\le \delta^{-(d-1)} \int_{|y''|\le C\delta} dy'' \int_{\mathbf{P}(0)\cap T^{C''\delta}_{\gamma_{(0,x_2)}}} |f(y_1, y_2, y'')|\, dy_1 dy_2 \\
&\lesssim \delta^{-(d-2)} \int_{|y''|\le C\delta} M_\delta(f(\,\cdot\,, y''))(x_2)\, dy'',
\end{aligned}$$

Therefore,

$$A(f)(x') \lesssim \delta^{-(d-2)} \int_{|y''|\le C\delta} M_\delta(f(\,\cdot\,, y''))(x_2)\, dy''.$$

Integrating over x_1, x_2 and using Minkowski's inequality, we get

$$\begin{aligned}
&\left(\int_{|x_2|\le 1} |A(f)(x')|^2 dx_2 \right)^{\frac{1}{2}} \\
&\lesssim \delta^{-(d-2)} \int_{|y''|\le C\delta} dy'' \left(\int_{|x_2|\le 1} |M_\delta(f(\,\cdot\,, y''))|^2(x_2)\, dx_2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}} \delta^{-(d-2)} \int_{|y''|\le C\delta} \|f(\,\cdot\,, y'')\|_{L^2(y_1,y_2)}\, dy'' \\
&\lesssim \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}} \delta^{-\frac{(d-2)}{2}} \|f\|_{L^2}.
\end{aligned}$$

Noticing $|x''| \lesssim \delta$ for $x \in V_k$, this leads to (1.3.6), so the proof is complete. $\square$

1.3.3 A key lemma

This section is parallel to section 2.4. From now on, let N be the number that fulfills both case I and $\mathrm{II}_{\theta,\sigma}$, and again we fix a index j such that $\boldsymbol{T}^\delta = T_j^\delta$ satisfy $\mathrm{II}_{\theta,\sigma}$. Using our L^2 estimate for the auxiliary maximal function, we will show that we can generalize Proposition 2.5 in [22] to any dimension $d \geq 3$.

Lemma 9. *For any $\epsilon > 0$, any point a*

$$|E \cap B(a, \delta^\epsilon \lambda)^c \cap \boldsymbol{T}^\sigma| \gtrsim_\epsilon \lambda^d N \sigma \delta^{d-2}. \tag{1.3.9}$$

Proof. Clearly, it suffices to prove

$$|E \cap \boldsymbol{T}^\sigma| \gtrsim_\epsilon \lambda^d N \sigma \delta^{d-2}. \tag{1.3.10}$$

For the tube $\boldsymbol{T}^\delta$, we denote

$$\boldsymbol{S}^\delta = \boldsymbol{T}^\delta \cap E \cap \left\{ x : \#\mathfrak{L}_{\theta,\sigma}(x,j) \geq 2^{-2} N \left(\log_2 \frac{1}{\delta^2}\right)^{-2} \right\}.$$

By the definition of $\mathfrak{L}_{\theta,\sigma}(x,j)$, we see that there is a $M_0 \in (0, M]$ and a subcollection $\{T_{i_k}^\delta\}_x$ of $\{T_i^\delta\}_{i=0}^M$ that are in $\mathfrak{L}_{\theta,\sigma}(x,j)$ for each x, if we let x run through every point in $\boldsymbol{S}^\delta$, and take the union of these subcollections to get $\{T_{i_k}^\delta\}_{k=1}^{M_0}$, then we will have

$$\sum_{k=1}^{M_0} \chi_{T_{i_k}^\delta} \geq \frac{N}{2^2} \left(\log_2 \frac{1}{\delta^2}\right)^{-2} \text{ on } \boldsymbol{S}^\delta.$$

It follows from Lemma 7 that two δ-tubes intersect at angle comparable to θ have intersection measure like $\frac{\delta^d}{\theta}$, so we have

$$|\boldsymbol{S}^\delta| \lesssim_\epsilon N^{-1} \int_{\boldsymbol{T}^\delta} \sum_{k=1}^{M_0} \chi_{T_{i_k}^\delta}(x) dx \leq N^{-1} \sum_{k=1}^{M_0} |T_{i_k}^\delta \cap \boldsymbol{T}^\delta| \lesssim \frac{M_0 \delta^d}{N\theta}.$$

Together with the simple fact

$$|\boldsymbol{S}^\delta| \gtrsim_\epsilon \lambda |\boldsymbol{T}^\delta|,$$

we conclude

$$M_0 \gtrsim_\epsilon \theta \delta^{-1} N \lambda. \tag{1.3.11}$$

Now, consider the average of the function $f = \chi_{E\cap\{y:\mathrm{dist}(y,\gamma_j)\in[\sigma/2,\sigma)\}}$ over $T^\delta_{i_k}$

$$\delta^{-(d-1)}\int_{T^\delta_{i_k}} f(y)\,dy = \delta^{-(d-1)}|T^\delta_{i_k}\cap E\cap\{y:\mathrm{dist}(y,\gamma_j)\in[\sigma/2,\sigma)\}| \gtrsim_\epsilon \lambda,$$

On the other hand, for some large C that only depends on curvature and some y'_{i_k} that lies in the $C_0\delta$-neighborhood of x'_{i_k}, we have

$$\delta^{-(d-1)}\int_{T^\delta_{i_k}} f(y)\,dy \lesssim A^{\theta,\sigma}_{C\delta}(f)(y'_{i_k}),$$

After combining these two inequalities, we square both sides, multiply both sides by δ^{d-1} and sum up with respect to $k = 1,\dots,M_0$, arriving at

$$\begin{aligned}
M_0\delta^{d-1}\lambda^2 &\lesssim_\epsilon \sum_{k=1}^{M_0}|A^{\theta,\sigma}_{C\delta}(f)(y'_{i_k})|^2\delta^{d-1}\\
&\lesssim \|A^{\theta,\sigma}_{C\delta}(f)(y')\|^2_{L^2}\\
&\lesssim_\epsilon \frac{\theta^{d-2}}{\sigma^{d-2}}|E\cap\{y:\mathrm{dist}(y,\gamma_j)\in[\sigma/2,\sigma)\}|\\
&\lesssim_\epsilon \frac{\theta}{\sigma\lambda^{d-3}}|E\cap\boldsymbol{T}^\sigma|,
\end{aligned}$$

where we used the maximality of the $\{x'_k\}$, (1.3.4) and (1.3.5). Using (1.3.11) for the estimate of M_0, we get (1.3.10). □

1.3.4 Completion of the proof

Again, we give the estimate corresponding to the high and low multiplicity cases separately.

As what happened in the Euclidean case, if N satisfy I, it's easy to see that

$$|E| \gtrsim \frac{\lambda M\delta^{d-1}}{N}. \tag{1.3.12}$$

In order to estimate the high multiplicity case, we need to use a curved version of the bush argument, which is the following lemma ([18]):

Lemma 10. *Suppose there are M tubes $\{T^\sigma_j\}_{j=1}^M$ such that $j\neq j'$ and $T^\sigma_j\cap T^\sigma_{j'}\neq\emptyset$ implies $\angle(T^\sigma_j,T^\sigma_{j'})\geq C\gamma$ for some $0<\gamma<1$. Assume also that for some $\rho>0$ and any $a\in\mathbb{R}^d$, there are M_0 such tubes satisfying*

$$\rho|T^\sigma_j| \leq |T^\sigma_j\cap E\cap B(a,\sigma/\gamma)^c|. \tag{1.3.13}$$

Then if C is large enough, we have

$$|E| \gtrsim \rho\sigma^{d-1}M_0^{1/2}. \tag{1.3.14}$$

By Lemma 7, the diameter of $T_{j'}^{\sigma} \cap T_j^{\sigma}$ is like σ/γ, thus the proof of this lemma is identical to that of Lemma 5.

Finally, we estimate the high multiplicity case to finish the proof.

Lemma 11. *Let N satisfy $\mathrm{II}_{\theta,\sigma}$, then we have*

$$|E| \gtrsim_{\epsilon} \lambda^{d+1}N(M\delta^{d-1})^{\frac{1}{d-1}}\delta^{d-2} \tag{1.3.15}$$

Proof. By the multiplicity argument, we know that for some suitable constant c, there are at least

$$[cM(\log_2 \frac{1}{\delta^2})^{-2}]$$

many tubes in $\mathrm{II}_{\theta,\sigma}$, denote them by

$$\{T_j^{\delta}\}_{j=1}^{[cM(\log_2 \frac{1}{\delta^2})^{-2}]}.$$

Let

$$\gamma = \frac{\sigma}{\delta^{\epsilon}\lambda}.$$

Then clearly $\gamma \geq \delta^{1-\epsilon}$. If $\gamma \geq 1$, then (1.3.15) follows directly from (1.3.9). Otherwise, take a maximal γ-separated subset of $\{x_j'\}_{j=1}^{[cM(\log_2 \frac{1}{\delta^2})^{-2}]}$ and denote the total number of this subset to be M_0. By maximality, we see easily

$$M_0 \gtrsim \frac{M}{\left(\log_2 \frac{1}{\delta^2}\right)^2}\delta^{d-1}\left(\frac{\delta^{\epsilon}\lambda}{\sigma}\right)^{d-1} \gtrsim_{\epsilon} M\delta^{d-1}\left(\frac{\lambda}{\sigma}\right)^{d-1}$$

and using (1.3.9) one may easily check that if we let $\rho = C_{\epsilon}\lambda^d\sigma^{2-d}\delta^{d-2+\epsilon}N$ for some proper constant

C_ϵ then all requirements of Lemma 10 are fulfilled, so we have

$$\begin{aligned}
|E| &\gtrsim_\epsilon \lambda^d \sigma^{2-d}\delta^{d-2}N \cdot \sigma^{d-1}M_0^{\frac{1}{2}} \\
&\geq \lambda^d \sigma \delta^{d-2} N M_0^{\frac{1}{d-1}} \\
&\gtrsim_\epsilon \lambda^d \sigma \delta^{d-2} N \left(M\delta^{d-1}\left(\frac{\lambda}{\sigma}\right)^{d-1}\right)^{\frac{1}{d-1}} \\
&= \lambda^d \sigma \delta^{d-2} N (M\delta^{d-1})^{\frac{1}{d-1}} \lambda \sigma^{-1} \\
&\geq \lambda^{d+1} N (M\delta^{d-1})^{\frac{1}{d-1}} \delta^{d-2},
\end{aligned}$$

where we used the fact that $M_0^{1/2} \geq M_0^{1/(d-1)}$ since $M_0 \geq 1$ and $d \geq 3$. $\square$

Now if we take the geometric mean of (1.3.11) and (1.3.15), we get (1.3.2), completing the proof of Theorem 2.

2

Improved geodesic restriction estimates for eigenfunctions

2.1 Introduction

The investigation of the eigenfunctions of the Laplace-Beltrami operator on Riemannian manifolds has been an ongoing endeavor for over one hundred years, and remains a central area in both mathematics and physics. Studying various types of concentration exhibited by eigenfunctions is essential in the development of this mathematical theory.

Let e_λ denote the L^2-normalized eigenfunction on a compact boundaryless manifold,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda,$$

so that λ is the eigenvalue of the first order operator $\sqrt{-\Delta_g}$.

It is a classical result of Sogge [20] that the L^p norms of the eigenfunctions satisfy

$$\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(M)}, \tag{2.1.1}$$

where $2 \le p \le \infty$ and $\sigma(p)$ is given by

$$\sigma(p) = \max\left\{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right),\ n\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}\right\}.$$

Alternatively, we can write

$$\|e_\lambda\|_{L^p(M)} \lesssim \begin{cases} \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}\|e_\lambda\|_{L^2(M)}, & 2 \le p \le \frac{2(n+1)}{n-1}, \\ \lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\|e_\lambda\|_{L^2(M)}, & \frac{2(n+1)}{n-1} \le p \le \infty. \end{cases} \tag{2.1.2}$$

Although the above estimates are sharp for the round sphere S^n due to various symmetries of the sphere, it is expected that one should be able to improve it for generic Riemannian manifolds.

Eigenfunctions on a manifold with nonpositive curvature have been studied actively as a model case. Indeed, in this setting, the eigenfunctions are conjectured to be distributed more and more evenly as the frequency $\lambda \to \infty$. The L^p norms of eigenfunctions are thus expected to be satisfying much better bounds than those in (2.1.1). It is a classical result of Bérard that one can get log improvements for sup-norms assuming nonpositive curvature, that is

$$\|e_\lambda\|_{L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}/\sqrt{\log \lambda}),$$

which gives log improvements over (2.1.1) for $p > p_c$ via interpolation.

Recently, log-type improvements over (2.1.1) for $2 < p < p_c$ and $p = p_c$ have been obtained by Blair-Sogge [4] and Sogge [24] respectively, essentially by proving log improved Kakeya-Nikodym bounds which measure L^2-concentration of eigenfunctions in $\lambda^{-\frac{1}{2}}$-neighborhoods about unit length geodesics.

In the last decade, similar L^p estimates have been established for the restriction of eigenfunctions to geodesics. Burq, Gérard and Tzvetkov [7] and Hu [13] showed that for n-dimensional Riemannian manifold (M, g), if $\mathit{\Pi}$ denotes the space of all unit-length geodesics γ, then

$$\sup_{\gamma \in \mathit{\Pi}} \Big(\int_\gamma |e_\lambda|^p \, ds \Big)^{\frac{1}{p}} \le C\lambda^{\sigma(n,p)}\|e_\lambda\|_{L^2(M)}, \tag{2.1.3}$$

where

$$\sigma(2,p) = \begin{cases} \frac{1}{4}, & 2 \le p \le 4, \\ \frac{1}{2} - \frac{1}{p}, & 4 \le p \le \infty. \end{cases} \tag{2.1.4}$$

and

$$\sigma(n,p) = \frac{n-1}{2} - \frac{1}{p}, \text{ if } p \ge 2 \text{ and } n \ge 3, \tag{2.1.5}$$

here the case $n = 3,\ p = 2$ is due to Chen and Sogge [9]. Note that in the 2-dimensional case, the estimates (2.1.3) have a similar flavor compared to Sogge's L^p estimates (2.1.1). Indeed, when $n = 2$, (2.1.3) also has a critical exponent $p_c = 4$. Moreover, on the round sphere S^2, (2.1.3) is saturated by zonal functions when $p \le 4$, while for $p \ge 4$, it is saturated by the highest weight spherical harmonics. When $n = 3$, the critical exponent no longer appears in (2.1.3). However, the estimate for $p = 2$ is still saturated by both zonal functions and highest weight spherical harmonics. In higher dimensions $n > 3$, geodesic restriction estimates are too singular to detect concentrations of eigenfunctions near geodesics. In fact, in these dimensions, estimates (2.1.3) are always saturated by zonal functions rather than highest weight spherical harmonics on the round sphere S^n.

There has been considerable work towards improving (2.1.3) under the assumption of nonpositive curvature in the 2-dimensional case. Bérard's sup-norm estimate [2] provides natural improvements for large p. In [8], Chen managed to improve over (2.1.3) for all $p > 4$ by a $(\log\lambda)^{-\frac{1}{2}}$ factor:

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^p\,ds\Big)^{\frac{1}{p}} \le C\frac{\lambda^{\frac{1}{2}-\frac{1}{p}}}{(\log\lambda)^{\frac{1}{2}}}\|e_\lambda\|_{L^2(M)}. \tag{2.1.6}$$

Sogge and Zelditch [25] showed that one can improve (2.1.3) for $2 \le p < 4$, in the sense that

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^p\,ds\Big)^{\frac{1}{p}} = o(\lambda^{\frac{1}{4}}). \tag{2.1.7}$$

A few years later, Chen and Sogge [9] showed that the same conclusion can be drawn for $p = 4$:

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^4\,ds\Big)^{\frac{1}{4}} = o(\lambda^{\frac{1}{4}}). \tag{2.1.8}$$

(2.1.8) was the first result to improve an estimate that is saturated by both zonal functions and highest weight spherical harmonics. Recently, by using the Toponogov's comparison theorem, Blair and Sogge [4] showed that it is possible to get log improvements for L^2-restriction:

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^2\,ds\Big)^{\frac{1}{2}} \le C\frac{\lambda^{\frac{1}{4}}}{(\log\lambda)^{\frac{1}{4}}}\|e_\lambda\|_{L^2(M)}, \tag{2.1.9}$$

In the joint work with Zhang, we obtained further improvements for the L^4-restriction estimates.

Theorem 5. *Let (M, g) be a 2-dimensional compact Riemannian manifold of nonpositive curvature, let $\gamma \subset M$ be a fixed unit-length geodesic segment. Then for $\lambda \gg 1$, there is a constant C such that*

$$\|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}f\|_{L^4(\gamma)} \le C\lambda^{\frac{1}{4}}(\log\log\lambda)^{-\frac{1}{8}}\|f\|_{L^2(M)}. \tag{2.1.10}$$

Therefore, taking $f = e_\lambda$, we have

$$\|e_\lambda\|_{L^4(\gamma)} \le C\lambda^{\frac14}(\log\log\lambda)^{-\frac18}\|e_\lambda\|_{L^2(M)}. \tag{2.1.11}$$

Moreover, if Π denotes the set of unit-length geodesics, there exists a uniform constant $C = C(M,g)$ such that

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^4\,ds\Big)^{\frac14} \le C\lambda^{\frac14}(\log\log\lambda)^{-\frac18}\|e_\lambda\|_{L^2(M)}. \tag{2.1.12}$$

Furthermore, if we assume further that M has constant negative curvature, we are able to get log improvement for the L^4-restriction estimate following the ideas in [4] and [9].

Theorem 6. *Let (M,g) be a 2-dimensional compact Riemannian manifold of constant negative curvature, let $\gamma \subset M$ be a fixed unit-length geodesic segment. Then for $\lambda \gg 1$, there is a constant C such that*

$$\|e_\lambda\|_{L^4(\gamma)} \le C\lambda^{\frac14}(\log\lambda)^{-\frac12}\|e_\lambda\|_{L^2(M)}. \tag{2.1.13}$$

Moreover, if Π denotes the set of unit-length geodesics, there exists a uniform constant $C = C(M,g)$ such that

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^4\,ds\Big)^{\frac14} \le C\lambda^{\frac14}(\log\lambda)^{-\frac12}\|e_\lambda\|_{L^2(M)}. \tag{2.1.14}$$

Remark 1. (2.1.14) is slightly better than the estimate originally stated in [30], however, as pointed out to us by Professor Sogge, this can be easily seen by a more careful analysis of the leading coefficient of the Hadamard parametrix.

This chapter is organized as follows. In Section 2.2, we give the proof of Theorem 5. We do this by first proving a new local restriction estimate which corresponds to Lemma 2.2 in [24]. Then we use this local estimate together with the improved L^2-restriction estimate (2.1.9) of Blair and Sogge [4] and the classical improved sup-norm estimate of Bérard [2] to obtain improved $L^2(M) \to L^{4,\infty}(\gamma)$ estimate. Finally, we prove Theorem 5 by interpolating between the improved $L^2(M) \to L^{4,\infty}(\gamma)$ estimate and the $L^2(M) \to L^{4,2}(\gamma)$ estimate of Bak and Seeger [1]. In Section 2.3, we show how to obtain further improvements under the assumption of constant negative curvature. We follow the strategies that were introduced in [9] and [4]. We shall lift all the calculations to the universal cover $\mathbb{H}^2$ and then use the Poincaré half-plane model to compute the dependence of various constants explicitly.

Throughout our argument, we shall assume that the injectivity radius of M is sufficiently large, say, larger than 10, and fix γ to be a unit length geodesic segment. We shall use P to denote the first order operator $\sqrt{-\Delta_g}$. Also, whenever we write $A \lesssim B$, it means $A \leq CB$ with C being some uniform constant depending only on the manifold.

2.2 Riemannian surface with nonpositive curvature

We start with some standard reductions. Let $\rho \in S(\mathbb{R})$ such that $\rho(0) = 1$ and supp $\hat{\rho} \subset [-1/2, 1/2]$, then it is clear that the operator $\rho(T(\lambda - P))$ reproduces eigenfunctions, in the sense that

$$\rho(T(\lambda - P))e_\lambda = e_\lambda.$$

Consequently, we would have the estimate (2.1.10) if we could show that

$$\|\rho(\log\lambda(\lambda - P))\|_{L^2(M)\to L^4(\gamma)} = O(\lambda^{\frac{1}{4}}/(\log\log\lambda)^{\frac{1}{8}}). \tag{2.2.1}$$

The uniform bound (2.1.12) also follows by a standard compactness argument.

2.2.1 A local restriction estimate

To prove (2.2.1), we apply Sogge's strategy in [24]. We shall need the following local restriction estimate.

Lemma 12. *Let $\lambda^{-1} \leq r \leq 1$, and γ_r be a fixed subsegment of γ with length r. Then we have*

$$\|\rho(\lambda - P)f\|_{L^2(\gamma_r)} \lesssim \lambda^{\frac{1}{4}} r^{\frac{1}{4}} \|f\|_{L^2(M)}.$$

Proof. By a standard TT^* argument, this is equivalent to showing that

$$\|\chi(\lambda - P)h\|_{L^2(\gamma_r)} \lesssim \lambda^{\frac{1}{2}} r^{\frac{1}{2}} \|h\|_{L^2(\gamma_r)}, \tag{2.2.2}$$

here $\chi = |\rho|^2$. Thus

$$\chi(\lambda - P)h = \frac{1}{2\pi}\int \hat{\chi}(t)e^{-i\lambda t}e^{itP}h\,dt.$$

We shall need a preliminary reduction. Let $\beta \in C_0^\infty$ be a Littlewood-Paley bump function, satisfying

$$\beta(s) = 1, \text{ if } s \in [1/2, 2], \quad \text{and } \beta(s) = 0, \text{ if } s \notin [1/4, 4].$$

Then we claim that it suffices to prove:

$$\Big\| \int \hat{\chi}(t) e^{-i\lambda t} \beta(P/\lambda) e^{itP} h \, dt \Big\|_{L^2(\gamma_r)} \le C \lambda^{\frac{1}{2}} r^{\frac{1}{2}} \|h\|_{L^2(\gamma_r)}. \tag{2.2.3}$$

Indeed, we note that the operator

$$\int \hat{\chi}(t) e^{-i\lambda t} (1 - \beta(P/\lambda)) e^{itP} \, dt \tag{2.2.4}$$

has kernel

$$\sum \hat{\chi}(\lambda - \lambda_j)(1-\beta)(\lambda_j/\lambda) e_j(\gamma(s)) \overline{e_j(\gamma(s'))}.$$

Since $\chi \in C_0^\infty(\mathbb{R})$ and β is the Littlewood-Paley bump function, we see that

$$|\hat{\chi}(\lambda - \lambda_j)(1-\beta)(\lambda_j/\lambda)| \le C(1 + \lambda + \lambda_j)^{-4}.$$

On the other hand, by the Weyl formula,

$$\sum_{\lambda_j \in [\lambda, \lambda+1]} |e_j(\gamma(s)) e_j(\gamma(s'))| \le C(1+\lambda),$$

we conclude that the kernel of the operator given by (2.2.4) is $O(\lambda^{-1})$. This means that this operator enjoys better bounds than (2.2.2), which gives our claim that it suffices to prove (2.2.3).

To prove (2.2.3), we consider the corresponding kernel

$$K_\lambda(\gamma(s), \gamma(s')) = \int \hat{\chi}(t) e^{-i\lambda t} \beta(P/\lambda) e^{itP}(\gamma(s), \gamma(s')) \, dt.$$

We claim that K_λ satisfies

$$|K_\lambda(\gamma(s), \gamma(s'))| = O(\lambda^{\frac{1}{2}} |s - s'|^{-\frac{1}{2}}). \tag{2.2.5}$$

Indeed, one may use a parametrix and the calculus of Fourier integral operators to see that modulo a trivial error term of size $O(\lambda^{-N})$

$$(\beta(P/\lambda) e^{itP})(\gamma(s), \gamma(s')) = \int_{\mathbb{R}^2} e^{i(s-s')\xi_1 + it|\xi|} \alpha(t, s, s', |\xi|) \, d\xi,$$

where α is a zero-order symbol. See the proof of [21, Lemma 5.1.3] and [23, Theorem 3.1.5]. Thus, modulo trivial errors,

$$K_\lambda(\gamma(s),\gamma(s')) = \int\int_0^\infty e^{it(l-\lambda)}\,\alpha(t,s,s',l)\,\Big(\int_{S^1} e^{il(s-s')\langle(0,1),\omega\rangle}\,d\omega\Big)\,l\,dldt. \tag{2.2.6}$$

Integrating by parts in t shows that the above expression is majorized by

$$\int_0^\infty (1+|l-\lambda|)^{-3} l\,dl = O(\lambda),$$

thus (2.2.5) is valid when $|s-s'|\le\lambda^{-1}$. To handle the remaining case, we recall that, by stationary phase,

$$\int_{S^1} e^{ix\cdot\omega}\,d\omega = O(|x|^{-\frac{1}{2}}), \quad |x|\ge 1.$$

If we plug this into (2.2.6) with $x = l(s-s',0)$, and integrate by parts in t, we conclude that if $\lambda^{-1}\le|s-s'|$, we have

$$|K_\lambda(\gamma(s),\gamma(s'))| \le \int_0^\infty (1+|l-\lambda|)^{-3}\,(l|s-s'|)^{-\frac{1}{2}}\,l\,dl = O(\lambda^{\frac{1}{2}}|s-s'|^{-\frac{1}{2}}),$$

as claimed. By Young's inequality, the left hand side of (2.2.3) is bounded by

$$\lambda^{\frac{1}{2}}\Big(\int_0^r\Big|\int_0^r \frac{1}{|s-s'|^{\frac{1}{2}}}\,h(s')\,ds'\Big|^2 ds\Big)^{\frac{1}{2}} \le \lambda^{\frac{1}{2}}r^{\frac{1}{2}}\|h\|_{L^2([0,r])},$$

completing our proof. □

Remark 2. In fact, Lemma 12 also follows from the L^4 restrction bound (2.1.3) and Cauchy-Schwarz inequality. However, we gave the proof above because a similar argument gives the same estimate for the more general operator $\rho(T(\lambda-P))f$ for all $T\ge 1$. Indeed, it is easy to see that the same proof works for operators with kernel of the form

$$\Big[\int a(t)\,e^{it\lambda}\,e^{-itP}\,dt\Big](\gamma(s),\gamma(s')), \tag{2.2.7}$$

providing $a\in C_0^\infty(-1,1)$. While the operator $\rho(T(\lambda-P))$ corresponds to the kernel

$$\Big[\frac{1}{T}\int a(t/T)\,e^{it\lambda}\,e^{-itP}\,dt\Big](\gamma(s),\gamma(s')),$$

which can be handled by smoothly partitioning the interval $[-T,T]$ into subintervals of size 1. Each piece of the kernel over a subinterval of size 1 enjoys the same bound as in Lemma 12, thanks to the fact that e^{-itP} is unitary on L^2. Now if we sum up the T pieces resulting from the partition, we obtain the desired estimate for $\rho(T(\lambda-P))$.

2.2.2 An improved weak-type estimate

In this section, we prove the following improved weak-type estimate.

Proposition 1. *Let (M,g) be a 2-dimensional compact Riemannian manifold of nonpositive curvature. Then for $\lambda \gg 1$*

$$\|\rho(\log\lambda(\lambda-P))\|_{L^2(M)\to L^{4,\infty}(\gamma)} = O(\lambda^{\frac{1}{4}}/(\log\log\lambda)^{\frac{1}{4}}). \tag{2.2.8}$$

As discussed before, the L^4 restriction bound is saturated by both zonal functions and highest weight spherical harmonics. Thus as in [24], to get improved L^4 bounds, we shall need the following improved results which corresponds to the range $2 \le p < 4$ and the range $4 < p \le \infty$ respectively.

Lemma 13 ([4]). *Let (M,g) be as above. Then for $\lambda \gg 1$ we have*

$$\|\rho(\log\lambda(\lambda-P))\|_{L^2(M)\to L^2(\gamma)} = O(\lambda^{\frac{1}{4}}/(\log\lambda)^{\frac{1}{4}}). \tag{2.2.9}$$

Lemma 14 ([2]). *If (M,g) is as above then there is a constant $C = C(M,g)$ so that for $T \ge 1$ and large λ we have the following bounds for the kernel of $\eta(T(\lambda-P))$, $\eta = \rho^2$,*

$$|\eta(T(\lambda-P))(x,y)| \le C\left[T^{-1}\Big(\frac{\lambda}{d_g(x,y)}\Big)^{\frac{1}{2}} + \lambda^{\frac{1}{2}}e^{CT}\right], \tag{2.2.10}$$

The first lemma is a recent result of Blair and Sogge [4]. The other bound (2.2.10) is well-known and follows from the arguments in the paper of Bérard [2].

Now we are ready to prove Proposition 1. It suffices to show that

$$|\{x\in\gamma : |\rho(\log\lambda(\lambda-P))f(x)| > \alpha\}| \le C\alpha^{-4}\lambda(\log\log\lambda)^{-1}. \tag{2.2.11}$$

assuming f is L^2 normalized. By Chebyshev inequality and (2.2.9), we have

$$\begin{aligned}|\{x\in\gamma : |\rho(\log\lambda(\lambda-P))f(x)| > \alpha\}| &\le \alpha^{-2}\int_\gamma |\rho(\log\lambda(\lambda-P))f|^2\,ds \\ &\le \alpha^{-2}\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{2}}.\end{aligned}$$

Note that for large λ we have

$$\alpha^{-2}\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{2}} \ll \alpha^{-4}\lambda(\log\log\lambda)^{-1}, \quad \text{if } \alpha \le \lambda^{\frac{1}{4}}(\log\lambda)^{\frac{1}{8}}.$$

Thus it remains to show

$$|\{x\in\gamma : |\rho(\log\lambda(\lambda-P))f(x)| > \alpha\}| \le C\alpha^{-4}\lambda(\log\log\lambda)^{-1},$$

when $\alpha \ge \lambda^{\frac{1}{4}}(\log\lambda)^{\frac{1}{8}}$.

We notice that

$$\big|\big[\rho(c_0\log\log\lambda(\lambda-\tau))-1\big]\rho(\log\lambda(\lambda-\tau))\big| \lesssim \frac{\log\log\lambda}{\log\lambda}(1+|\lambda-\tau|)^{-N},$$

together with the estimate

$$\|\chi_\lambda\|_{L^2(M)\to L^4(\gamma)} = O(\lambda^{\frac{1}{4}}),$$

we see that

$$\big\|\big[\rho(c_0\log\log\lambda(\lambda-P))-I\big]\circ\rho(\log\lambda(\lambda-P))f\big\|_{L^4(\gamma)} \lesssim \frac{\log\log\lambda}{\log\lambda}\lambda^{\frac{1}{4}}\|f\|_{L^2(M)}.$$

Therefore we would be done if we could show that

$$|\{x\in\gamma : |\rho(c_0\log\log\lambda(\lambda-P))h(x)| > \alpha\}| \le C\alpha^{-4}\lambda(\log\log\lambda)^{-1},$$

if $\alpha \ge \lambda^{\frac{1}{4}}(\log\lambda)^{\frac{1}{8}}$, and $\|h\|_{L^2(M)} = 1$.

Let

$$A = \{x\in\gamma : |\rho(c_0\log\log\lambda(\lambda-P))h(x)| > \alpha\}.$$

Take

$$r = \lambda\alpha^{-4}(\log\log\lambda)^{-2}.$$

We decompose A into r-separated subsets $\cup_j A_j = A$ with length $\approx r$. By replacing A by a set of proportional measure, we may assume that if $j\ne k$, we have $\mathrm{dist}(A_j,A_k) > C_0 r$, where C_0 will be specified momentarily.

Let $T_\lambda = \rho(c_0\log\log\lambda(\lambda-P))$, which has dual operator T_λ^* mapping $L^2(\gamma)\to L^2(M)$. Let $\psi_\lambda(x) = T_\lambda f(x)/|T_\lambda f(x)|$, if $T_\lambda f(x)\ne 0$, otherwise let $\psi_\lambda(x)=1$. Let $S_\lambda = T_\lambda T_\lambda^*$ and $a_j = \overline{\psi_\lambda 1_{A_j}}$.

Then by Chebyshev's inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\alpha|A| &\le \left|\int_\gamma T_\lambda f\overline{\psi_\lambda 1_A}\,ds\right| \le \left|\int_\gamma \sum_j T_\lambda f a_j\,ds\right| \\ &= \left|\int_M \sum_j T_\lambda^* a_j f\,dV_g\right| \le \left(\int_M \Big|\sum_j T_\lambda^* a_j\Big|^2 dV_g\right)^{\frac12},\end{aligned}$$

squaring both sides, we see that

$$\alpha^2|A|^2 \le \sum_j \int_M |T_\lambda^* a_j|^2 dV_g + \sum_{j\neq k}\int_\gamma S_\lambda a_j\overline{a_k}\,ds = I + II.$$

By the dual version of Lemma 12 (see Remark 2), we see that

$$I \lesssim r^{\frac12}\lambda^{\frac12}\sum_j\int_\gamma |a_j|^2\,ds = r^{\frac12}\lambda^{\frac12}|A| = \lambda\alpha^{-2}(\log\log\lambda)^{-1}|A|.$$

By making c_0 sufficiently small, we see from (2.2.10) that we can control the kernel, $K_\lambda(s,s')$, of S_λ by

$$|K_\lambda(s,s')| \le C\Big[(\log\log\lambda)^{-1}\Big(\frac{\lambda}{|s-s'|}\Big)^{\frac12} + \lambda^{\frac12}(\log\lambda)^{\frac{1}{40}}\Big],$$

thus

$$\begin{aligned}II &\lesssim \Big[(\log\log\lambda)^{-1}\Big(\frac{\lambda}{C_0 r}\Big)^{\frac12} + \lambda^{\frac12}(\log\lambda)^{\frac{1}{40}}\Big]\sum_{j\neq k}\|a_j\|_{L^1}\|a_k\|_{L^1} \\ &\le C_0^{-\frac12}\alpha^2|A|^2 + \lambda^{\frac12}(\log\lambda)^{\frac{1}{40}}|A|^2.\end{aligned}$$

Since we are assuming $\alpha \ge \lambda^{\frac14}(\log\lambda)^{\frac18}$, for sufficiently large C_0, we have

$$II \le \frac12\alpha^2|A|^2,$$

thus

$$\alpha^2|A|^2 \le C\lambda\alpha^{-2}(\log\log\lambda)^{-1}|A| + \frac12\alpha^2|A|^2,$$

which gives

$$|A| \le C\lambda\alpha^{-4}(\log\log\lambda)^{-1}, \quad \text{if } \alpha \ge \lambda^{\frac14}(\log\lambda)^{\frac18},$$

completing the proof of Proposition 1.

2.2.3 Proof of Theorem 5

We shall combine the improved $L^2(M) \to L^{4,\infty}(\gamma)$ estimate (2.2.8) we obtained in the last section with the following $L^2(M) \to L^{4,2}(\gamma)$ estimate established by Bak and Seeger [1] to prove our main theorem. This estimate of Bak and Seeger holds for general Riemannian manifold without any curvature condition.

Lemma 15 ([1])**.** *Let (M,g) be a 2-dimensional Riemannian manifold. Fix $\gamma \subset M$ to be a geodesic segment. Then we have the following estimate for the unit band spectral projection operator $\chi_{[\lambda,\lambda+1]}$*

$$\|\chi_{[\lambda,\lambda+1]} f\|_{L^{4,2}(\gamma)} \le C(1+\lambda)^{\frac{1}{4}} \|f\|_{L^2(M)}. \tag{2.2.12}$$

We remark that Lemma 15 is a special case of the results in [1] regarding the restriction of eigenfunctions to hypersurfaces for manifolds with dimension $n \ge 2$.

Let us recall some basic facts about the Lorentz space $L^{p,q}(\gamma)$. First, for a function u on M, we define the corresponding distribution function $\omega(\alpha)$ with respect to γ as

$$\omega(\alpha) = |\{x \in \gamma : |u(x)| > \alpha\}, \quad \alpha > 0.$$

Let u^* be the nonincreasing rearrangement of u on γ, given by

$$u^*(t) = \inf\{\alpha : \omega(\alpha) \le t\}, \quad t > 0.$$

Then the Lorentz space $L^{p,q}(\gamma)$ for $1 \le p < \infty$ and $1 \le q < \infty$ is defined as all u so that

$$\|u\|_{L^{p,q}(\gamma)} = \Big(\frac{q}{p}\int_0^\infty \big[t^{\frac{1}{p}} u^*(t)\big]^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty, \tag{2.2.13}$$

It is well known that for the special case $p = q$, the Lorentz norm $\|\cdot\|_{L^{p,p}(\gamma)}$ agrees with the standard L^p norm $\|\cdot\|_{L^p(\gamma)}$. Moreover, we also have

$$\sup_{t>0} t^{\frac{1}{p}} u^*(t) = \sup_{\alpha>0} \alpha[\omega(\alpha)]^{\frac{1}{p}}.$$

If we take $u = \chi_{[\lambda,\lambda+(\log\lambda)^{-1}]} f$, and assume $\|f\|_{L^2(M)} = 1$, then by (2.2.8) we have

$$\sup_{t>0} t^{\frac{1}{4}} u^*(t) \le C\lambda^{\frac{1}{4}} (\log\log\lambda)^{-\frac{1}{4}}. \tag{2.2.14}$$

On the other hand, since $\chi_{[\lambda,\lambda+1]}u = u$, by Lemma 15 we have

$$\|u\|_{L^{4,2}(\gamma)} \le C\lambda^{\frac{1}{4}}\|u\|_{L^2(M)} \le C\lambda^{\frac{1}{4}}. \tag{2.2.15}$$

Interpolating between (2.2.14) and (2.2.15), we then get

$$\begin{aligned}\|u\|_{L^4(\gamma)} &= \left(\int_0^\infty \left[t^{\frac{1}{4}}u^*(t)\right]^4 \frac{dt}{t}\right)^{\frac{1}{4}}\\ &\lesssim \left(\sup_{t>0} t^{\frac{1}{4}}u^*(t)\right)^{\frac{1}{2}} \|u\|_{L^{4,2}(\gamma)}^{\frac{1}{2}}\\ &\lesssim \left(\lambda^{\frac{1}{4}}(\log\log\lambda)^{-\frac{1}{4}}\right)^{\frac{1}{2}}\lambda^{\frac{1}{8}}\\ &= \lambda^{\frac{1}{4}}(\log\log\lambda)^{-\frac{1}{8}},\end{aligned}$$

which completes the proof of Theorem 5.

2.3 Riemannian surfaces with constant negative curvature

We shall apply the strategies in [9] and [4] to prove Theorem 6. Recall that in [9], Chen and Sogge showed that for Riemannian surfaces with nonpositive curvature,

$$\begin{aligned}&\left(\int_0^1 \left|\int_0^1 \chi(T(\lambda-P))(\gamma(t),\gamma(s))h(s)ds\right|^4 dt\right)^{\frac{1}{4}}\\ &\le CT^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\|h\|_{L^{\frac{4}{3}}([0,1])} + C_T\lambda^{\frac{3}{8}}\|h\|_{L^{\frac{4}{3}}([0,1])},\end{aligned} \tag{2.3.1}$$

here $\chi(T(\lambda-P))(x,y)$ denotes the kernel of the multiplier operator $\chi(T(\lambda-P))$. Clearly, this would imply (2.1.8) if one takes T to be sufficiently large. We shall show that under the assumption of constant negative curvature, the constant C_T in (2.3.1) can be taken to be e^{CT} where $C > 0$ is some constant independent of T. Then if we set $T = c\log\lambda$, for some small $c > 0$, we can obtain log improvements. From now on, we shall use C to denote various positive constants that are independent of T and λ.

2.3.1 Some reductions

Choose a bump function $\beta \in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(\tau) = 1 \quad \text{for} \quad |\tau| \le 3/2, \quad \text{and} \quad \beta(\tau) = 0, \quad |\tau| \ge 2.$$

Then we may write

$$\chi(T(\lambda-P))(x,y)=\frac{1}{2\pi T}\int\beta(\tau)\hat{\chi}(\tau/T)e^{i\lambda\tau}(e^{-i\tau P})(x,y)\,d\tau$$
$$+\frac{1}{2\pi T}\int(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(e^{-i\tau P})(x,y)\,d\tau=K_0(x,y)+K_1(x,y).$$

As (2.2.5) in the proof of Lemma 1, one may use a parametrix to see that

$$\left(\int_0^1\left|\int_0^1 K_0(\gamma(t),\gamma(s))h(s)ds\right|^4 dt\right)^{\frac{1}{4}}\leq CT^{-1}\lambda^{\frac{1}{2}}\|h\|_{L^{\frac{4}{3}}([0,1])},\tag{2.3.2}$$

which is better than the bounds in (2.3.1). (See [9, p.8].) Since the kernel of $\chi(T(\lambda+P))$ is $O(\lambda^{-N})$ with constants independent of T, we are left to consider the integral operator S_λ:

$$S_\lambda h(t)=\frac{1}{\pi T}\int_{-\infty}^{\infty}\int_0^1(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(\cos\tau P)(\gamma(t),\gamma(s))h(s)\,dsd\tau.\tag{2.3.3}$$

As in [9] and [4], we now use the Hadamard parametrix and the Cartan-Hadamard theorem to lift the calculations up to the universal cover $(\mathbb{H}^2,\tilde{g})$ of (M,g).

Let Γ denote the group of deck transformations preserving the associated covering map $\kappa:\mathbb{H}^2\to M$ coming from the exponential map from $\gamma(0)$ associated with the metric g on M. The metric $\tilde{g}$ is the usual metric on $\mathbb{H}^2$ for the upper half plane model. Choose also a Dirchlet fundamental domain, $D\simeq M$, for M centered at the lift $\tilde{\gamma}(0)$ of $\gamma(0)$. We shall let $\tilde{\gamma}(t)$ denote the lift of the geodesic $\gamma(t)$, containing the unit geodesic segment $\gamma(t),t\in[0,1]$. We measure the distances in $\mathbb{H}^2$ using its Riemannian distance function $d_{\tilde{g}}(\,\cdot\,,\,\cdot\,)$.

Following [9], we recall that if $\tilde{x}$ denotes the lift of $x\in M$ to D, then we have the following formula

$$(\cos t\sqrt{-\Delta_g})(x,y)=\sum_{\alpha\in\Gamma}(\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x},\alpha(\tilde{y})).$$

Consequently, we have, for $t\in[0,1]$,

$$S_\lambda h(t)=\frac{1}{\pi T}\sum_{\alpha\in\Gamma}\int_{-\infty}^{\infty}\int_0^1(1-\beta(\tau))\hat{\chi}(\tau/T)e^{i\lambda\tau}(\cos\tau\sqrt{-\Delta_{\tilde{g}}})(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))h(s)\,dsd\tau.$$

Let

$$\mathrm{T}_R(\tilde{\gamma})=\{(x,y)\in\mathbb{R}^2:d_{\tilde{g}}((x,y),\tilde{\gamma})\leq R\}\tag{2.3.4}$$

and

$$\Gamma_{\mathrm{T}_R(\tilde{\gamma})} = \{\alpha \in \Gamma : \alpha(D) \cap \mathrm{T}_R(\tilde{\gamma}) \neq \emptyset\}.$$

From now on we fix $R \approx \mathrm{Inj}M$.

We write

$$S_\lambda h(t) = S_\lambda^{tube} h(t) + S_\lambda^{osc} h(t) = \sum_{\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}} S_\lambda^\alpha h(t) + \sum_{\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}} S_\lambda^\alpha h(t), t \in [0,1].$$

By the Huygens principle,

$$(\cos \tau \sqrt{-\Delta_{\tilde{g}}})(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) = 0, \quad \text{if} \quad d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) > \tau.$$

Recall that $\hat{\chi}(\tau) = 0$ if $|\tau| \geq 1$. Hence $d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) \leq T, s, t \in [0,1]$.

Since there are only $O(1)$ "translates" of D, $\alpha(D)$, that intersect any geodesic ball with arbitrary center of radius R, it follows that

$$\#\{\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})} : d_{\tilde{g}}(0, \alpha(0)) \in [2^k, 2^{k+1}]\} \leq C2^k. \tag{2.3.5}$$

Thus the number of nonzero summands in $S_\lambda^{tube} h(t)$ is $O(T)$ and in $S_\lambda^{osc} h(t)$ is $O(e^{CT})$.

Given $\alpha \in \Gamma$ set with $s, t \in [0,1]$

$$K_\alpha(t,s) = \frac{1}{\pi T} \int_{-T}^{T} (1 - \beta(\tau)) \hat{\chi}(\tau/T) e^{i\lambda\tau} (\cos \tau \sqrt{-\Delta_{\tilde{g}}})(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))\, d\tau.$$

When $\alpha = Identity$, by using the Hadamard parametrix (see e.g. [8, p. 9]), we get

$$|K_{\mathrm{Id}}(t,s)| \leq CT^{-1} \lambda^{\frac{1}{2}} |t-s|^{-\frac{1}{2}}.$$

Thus, by Hardy-Littlewood-Sobolev inequality, the corresponding operator is bounded from $L^{\frac{4}{3}}([0,1])$ to $L^4([0,1])$ with norm $CT^{-1}\lambda^{\frac{1}{2}}$.

If $\alpha \neq Identity$, we set

$$\phi(t,s) = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))), \quad s, t \in [0,1].$$

Then by the Huygens principle and the fact that $\alpha \neq Identity$, we have

$$2 \leq \phi(t,s) \leq T, \quad \text{if} \quad s,t \in [0,1]. \tag{2.3.6}$$

Following Lemma 3.1 in [9], we can write

$$K_\alpha(t,s) = w(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) \sum_{\pm} a_\pm(T, \lambda; \phi(t,s)) e^{\pm i\lambda\phi(t,s)} + R(t,s),$$

where $|w(x,y)| \leq Ce^{-cd_{\tilde{g}}(x,y)}$ by the Gunther comparison theorem, and for each $j = 0, 1, 2, ...,$ there is a constant C_j independent of T, $\lambda \geq 1$ so that

$$\left|\partial_r^j a_\pm(T, \lambda; r)\right| \leq C_j T^{-1} \lambda^{\frac{1}{2}} r^{-\frac{1}{2}-j}, r \geq 1. \tag{2.3.7}$$

Using the Hadamard parametrix with an estimate on the remainder term (see [23]), we see that

$$|R(t,s)| \leq e^{CT}.$$

Therefore we are able to estimate $S_\lambda^\alpha h$, $\alpha \neq \mathrm{Id}$ by Young's inequality. Indeed, the kernel satisfies

$$\left| \sum_{\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})},\, \alpha \neq \mathrm{Id}} K_\alpha(t,s) \right| \leq CT^{-1}\lambda^{\frac{1}{2}} \sum_{1 \leq 2^k \leq T} e^{-c2^k} 2^k 2^{-k/2} + e^{CT} = CT^{-1}\lambda^{\frac{1}{2}} + e^{CT}.$$

Consequently,

$$\left\|S_\lambda^{tube} h\right\|_{L^4([0,1])} \leq (CT^{-1}\lambda^{\frac{1}{2}} + e^{CT}) \|h\|_{L^{\frac{4}{3}}([0,1])}. \tag{2.3.8}$$

2.3.2 A stationary phase argument

To deal with the remaining part $S_\lambda^{osc} h(t)$, we need the following detailed version of the oscillatory integral estimates. (See e.g. [21, Chapter 1]).

Proposition 2. *Let $a \in C_0^\infty(\mathbb{R}^2)$, let $\phi \in C^\infty(\mathbb{R}^2)$ be real valued and $\lambda > 0$, set*

$$T_\lambda f(t) = \int_{-\infty}^{\infty} e^{i\lambda\phi(t,s)} a(t,s) f(s)\, ds, \quad f \in C_0^\infty(\mathbb{R}).$$

If $\phi_{st}^{''} \neq 0$ on $\operatorname{supp} a$, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C_{a,\phi} \lambda^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C_{a,\phi} = C\text{diam}(\text{supp } a)^{\frac{1}{2}} \left\{ \|a\|_\infty + \frac{\sum_{0\leq i,j\leq 2} \|\partial_t^i a\|_\infty \|\partial_t^j \phi_{st}^{''}\|_\infty}{\inf |\phi_{st}^{''}|^2} \right\}. \tag{2.3.9}$$

Assume that there is one t_0 such that $\phi_{st}^{''}(t_0, s) = 0$, and $\phi_{stt}^{'''}(t_0, s) \neq 0$ for all $(t_0, s) \in \operatorname{supp} a$, and $\phi_{st}^{''}(t, s) \neq 0$ for all $t \neq t_0$, then

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C'_{a,\phi} \lambda^{-\frac{1}{4}} \|f\|_{L^2(\mathbb{R})},$$

where

$$C'_{a,\phi} = C\text{diam}(\text{supp } a)^{\frac{1}{4}} \left\{ \|a\|_\infty + \frac{\sum_{0\leq i,j\leq 2} \|\partial_t^i a\|_\infty \|\partial_t^j \phi_{st}^{''}\|_\infty}{\inf |\phi_{st}^{''}/(t-t_0)|^2} \right\}. \tag{2.3.10}$$

Here the infimums are taken on supp *a.*

Proof. By a standard TT^* argument and Young's inequality, it suffices to estimate the kernel of $T_\lambda^* T_\lambda$

$$K(s, s') = \int e^{i\lambda(\phi(t,s)-\phi(t,s'))} a(t, s)\overline{a(t, s')}\, dt.$$

Let

$$\varphi(t, s, s') = \frac{\phi(t, s) - \phi(t, s')}{s - s'},\ s \neq s',\ \text{and}\ \varphi(t, s, s) = \phi'_s(t, s),$$

and let

$$\tilde{a}(t, s, s') = a(t, s)\overline{a(t, s')}.$$

Then the kernel reads

$$K(s, s') = \int e^{i\lambda(s-s')\varphi(t,s,s')} \tilde{a}(t, s, s')\, dt. \tag{2.3.11}$$

If $\phi_{st}^{''} \neq 0$ on $\operatorname{supp} a$, then by the mean value theorem,

$$|\varphi'_t(t, s, s')| = |\phi''_{st}(t, s'')| \geq \inf|\phi''_{st}|,$$

where s'' is some number between s and s'. If $\lambda(s - s') \leq 1$, it is easy to see that

$$|K(s, s')| \leq \int |a(t, s)||a(t, s')|\, dt \leq \text{diam}(\operatorname{supp} a)\|a\|_\infty^2.$$

For $\lambda(s-s')\geq 1$, we integrate by parts twice to see that

$$\begin{aligned}|K(s,s')| &\leq (\lambda|s-s'|)^{-2}\int\left|\frac{\partial}{\partial t}\left(\frac{1}{\varphi_t'}\frac{\partial}{\partial t}\left(\frac{\tilde{a}}{\varphi_t'}\right)\right)\right|dt\\ &\leq C(\lambda|s-s'|)^{-2}\text{diam}(\text{supp } a)\frac{\left(\sum\limits_{0\leq i,j\leq 2}||\partial_t^i a||_\infty||\partial_t^j\phi_{st}''||_\infty\right)^2}{\inf|\phi_{st}''|^4},\end{aligned}$$

where C is some uniform constant.

Hence

$$|K(s,s')|\leq C\text{diam}(\text{supp } a)\left\{\|a\|_\infty^2+\frac{\left(\sum\limits_{0\leq i,j\leq 2}||\partial_t^i a||_\infty||\partial_t^j\phi_{st}''||_\infty\right)^2}{\inf|\phi_{st}''|^4}\right\}(1+\lambda|s-s'|)^{-2},$$

again C is some constant independent of λ, a and ϕ. Consequently,

$$\int|K(s,s')|ds\leq C_{a,\phi}^2\lambda^{-1},$$

which finishes the proof of the first case.

Now we prove the second part of our proposition. Let $\delta>0$. Choose $\rho\in C_0^\infty(\mathbb{R})$ satisfying $\rho(t)=1$, $|t|\leq 1$, and $\rho(t)=0$, $|t|\geq 2$. Then

$$\left|\int e^{i\lambda(s-s')\varphi}\tilde{a}\rho((t-t_0)/\delta)\,dt\right|\leq 4\delta\|a\|_\infty^2.$$

For the remainder term with factor $1-\rho$, we integrate by parts twice to see that if $s\neq s'$,

$$\begin{aligned}&\left|\int e^{i\lambda(s-s')\varphi}\tilde{a}(1-\rho((t-t_0)/\delta))\,dt\right|\\ &\leq(\lambda|s-s'|)^{-2}\int\limits_{|t-t_0|>\delta}\left|\frac{\partial}{\partial t}\left(\frac{1}{\varphi_t'}\frac{\partial}{\partial t}\left(\frac{\tilde{a}(1-\rho((t-t_0)/\delta))}{\varphi_t'}\right)\right)\right|dt\\ &\leq C(\lambda|s-s'|)^{-2}\frac{\left(\sum\limits_{0\leq i,j\leq 2}||\partial_t^i a||_\infty||\partial_t^i\phi_{st}''||_\infty\right)^2}{\inf\left(|\phi_{st}''|/|t-t_0|\right)^4}\int\limits_{|t-t_0|>\delta}(|t-t_0|^{-4}+\delta^{-2}|t-t_0|^{-2})\,dt\\ &\leq C\delta^{-3}(\lambda|s-s'|)^{-2}\frac{\left(\sum\limits_{0\leq i,j\leq 2}||\partial_t^i a||_\infty||\partial_t^i\phi_{st}''||_\infty\right)^2}{\inf\left(|\phi_{st}''|/|t-t_0|\right)^4},\end{aligned}$$

where C is a constant independent of λ, a, ϕ and F.

By setting $\delta = (\lambda|s-s'|)^{-\frac{1}{2}}$, we get

$$|K(s,s')| \leq C\left\{\|a\|_\infty^2 + \frac{\left(\sum_{0\leq i,j\leq 2}||\partial_t^i a||_\infty||\partial_t^j \phi_{st}^{''}||_\infty\right)^2}{\inf(|\phi_{st}^{''}|/|t-t_0|)^4}\right\}(\lambda|s-s'|)^{-\frac{1}{2}}, \quad \text{if } s \neq s'.$$

Therefore,

$$\int |K(s,s')|ds \leq C_{a,\phi}^{'2}\lambda^{-\frac{1}{2}},$$

which completes the proof. □

2.3.3 Proof of Theorem 6

Noting that diam(supp $a_\pm$) ≤ 2 and we have good control on the size of $a_\pm$ and its derivatives by (2.3.7), it remains to estimate the size of $\phi_{st}^{''}$ and its derivatives. On general surfaces with nonpositive curvature, it seems difficult to get desirable bounds. However, under our assumption of constant curvature, we can compute $\phi_{st}^{''}$ and its derivatives explicitly.

Without loss of generality, we may assume that (M,g) is a compact Riemannian surface with constant curvature equal to -1. It is well known that the universal cover of any Riemannian surface with constant negative curvature -1 is the hyperbolic plane $\mathbb{H}^2$. We consider the Poincaré half-plane model

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\},$$

with the metric given by

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

Recall that the distance function for the Poincaré half-plane model is given by

$$\mathrm{dist}((x_1,y_1),(x_2,y_2)) = \operatorname{arcosh}\left(1 + \frac{(x_2-x_1)^2 + (y_2-y_1)^2}{2y_1y_2}\right),$$

where arcosh is the inverse hyperbolic cosine function

$$\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2-1}),\ x \geq 1.$$

Moreover, the geodesics are the straight vertical rays orthogonal to the x-axis and the half-circles whose centers are on the x-axis. Any pair of geodesics can intersect at at most one point. Without loss of generality, we may assume that $\tilde{\gamma}$ is the y-axis. There are three possibilities for the image

$\alpha(\tilde{\gamma})$. It can be a straight line parallel to $\tilde{\gamma}$, a half-circle parallel to $\tilde{\gamma}$, or a half-circle intersecting $\tilde{\gamma}$ at one point. We need to treat these cases separately.

Let $\{\tilde{\gamma}(t) = (0, e^t),\ t \in \mathbb{R}\}$ be the infinite geodesic parameterized by arclength. Our unit geodesic segment is given by $\{\tilde{\gamma}(t),\ t \in [0,1]\}$. Then its image $\{\alpha(\tilde{\gamma}(s)),\ s \in [0,1]\}$, is a unit geodesic segment of $\alpha(\tilde{\gamma})$.

Lemma 16. *If $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$ and $\alpha(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset$, then we have*

$$\inf\ |\phi_{st}^{''}| \geq e^{-CT},$$

and

$$\|\phi_{st}^{''}\|_\infty + \|\phi_{stt}^{'''}\|_\infty + \|\phi_{sttt}^{''''}\|_\infty \leq e^{CT},$$

where $C > 0$ is independent of T. The infimum and the norm are taken on the unit square $\{(t,s) \in \mathbb{R}^2 : t, s \in [0,1]\}$.

Lemma 17. *Let $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$ and $\alpha(\tilde{\gamma})$ be a half-circle intersecting $\tilde{\gamma}$ at the point $(0, e^{t_0}),\ t_0 \in \mathbb{R}$.*

If $t_0 \notin [-1,2]$, then the intersection point $(0, e^{t_0})$ is outside the geodesic segment $\{\tilde{\gamma}(t) : t \in [-1,2]\}$. We have

$$\inf\ |\phi_{st}^{''}| \geq e^{-CT},$$

and

$$\|\phi_{st}^{''}\|_\infty + \|\phi_{stt}^{'''}\|_\infty + \|\phi_{sttt}^{''''}\|_\infty \leq e^{CT},$$

where $C > 0$ is independent of T.

On the other hand, if $t_0 \in [-1,2]$, we have

$$\inf\ |\phi_{st}^{''}/(t - t_0)| \geq e^{-CT},$$

and

$$\|\phi_{st}^{''}\|_\infty + \|\phi_{stt}^{'''}\|_\infty + \|\phi_{sttt}^{''''}\|_\infty \leq e^{CT},$$

where $C > 0$ is independent of T. The infima and the norms are taken on the unit square $\{(t,s) \in \mathbb{R}^2 : t, s \in [0,1]\}$.

We shall postpone the proof of Lemma 16 and Lemma 17 to the last section. Now we see first how to finish the proof of Theorem 6 using Lemma 16 and Lemma 17.

Proof of Theorem 6. By (2.3.2) and (2.3.8), we only need to show that

$$\|S_\lambda^{osc} h\|_{L^4([0,1])} \le e^{CT} \lambda^{\frac{3}{8}} \|h\|_{L^{\frac{4}{3}}([0,1])}, \tag{2.3.12}$$

where C is independent of T.

Let $\alpha \notin \Gamma_{\mathrm{T}_R}(\tilde{\gamma})$. If $\alpha(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset$, by Proposition 2, Lemma 16 and the condition on the amplitude (2.3.7), we have

$$\|S_{\lambda,\alpha}^{osc} h\|_{L^2([0,1])} \le e^{CT} \|h\|_{L^2([0,1])}.$$

Assume that $\alpha(\tilde{\gamma})$ intersects $\tilde{\gamma}$ at the point $\tilde{\gamma}(t_0)$. Since $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$, the intersection point cannot lie on the unit geodesic segment $\alpha(\tilde{\gamma}(s)),\ s \in [0,1]$. Thus, by Proposition 2, Lemma 17 and (2.3.7) we obtain

$$\|S_{\lambda,\alpha}^{osc} h\|_{L^2([0,1])} \le e^{CT} \|h\|_{L^2([0,1])}, \text{ if } t_0 \notin [-1,2],$$

and

$$\|S_{\lambda,\alpha}^{osc} h\|_{L^2([0,1])} \le e^{CT} \lambda^{\frac{1}{4}} \|h\|_{L^2([0,1])}, \text{ if } t_0 \in [-1,2],$$

where we set $F = [-1,2] \times [0,1]$ in Proposition 2 for the case $t_0 \in [-1,2]$.

Recall that the number of nonzero summands in S_λ^{osc} is $O(e^{CT})$. Consequently, for $\lambda > 1$ we always have

$$\|S_\lambda^{osc} h\|_{L^2([0,1])} \le e^{CT} \lambda^{\frac{1}{4}} \|h\|_{L^2([0,1])}.$$

By interpolating with the trivial $L^1 \to L^\infty$ bound, we obtain (2.3.12), finishing the proof. $\square$

2.3.4 Proof of Lemmas

Before proving the lemmas, we remark that in the Poincaré half-plane model

$$\mathrm{T}_R(\tilde{\gamma}) = \{(x,y) \in \mathbb{R}^2 : y > 0 \text{ and } y \ge |x| / \sqrt{(\cosh R)^2 - 1}\}.$$

Indeed, the distance between $(0, e^t)$ and (x,y), $y > 0$, is

$$f(t) = \operatorname{arcosh}\Big(1 + \frac{x^2 + (y - e^t)^2}{2ye^t}\Big) = \operatorname{arcosh}\Big(\frac{x^2 + y^2 + e^{2t}}{2ye^t}\Big).$$

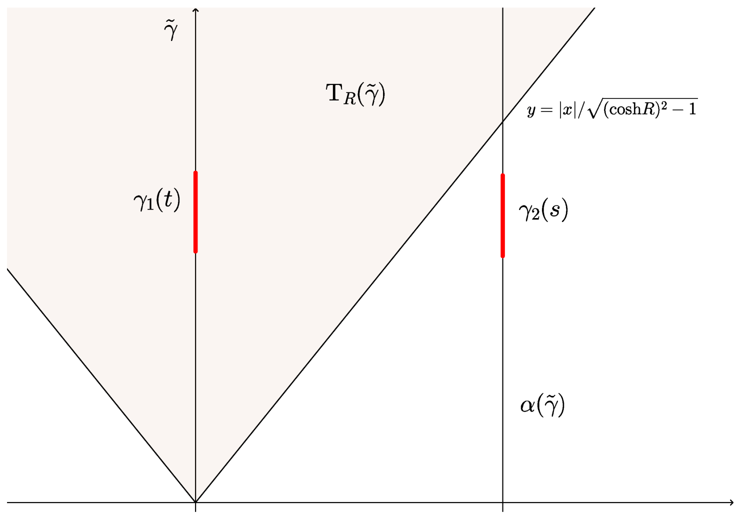

Figure 2.1: $\alpha(\tilde{\gamma})$ is a line parallel to $\tilde{\gamma}$.

Setting $f'(t) = 0$ gives $t = \ln\sqrt{x^2 + y^2}$, which must be the only minimum point. Thus the distance between (x, y) and the infinite geodesic $\tilde{\gamma}$ is

$$\mathrm{dist}((x, y), \tilde{\gamma}) = \mathrm{arcosh}(\sqrt{1 + (x/y)^2}).$$

Since $\mathrm{dist}((x, y), \tilde{\gamma}) \leq R$ in $\mathrm{T}_R(\tilde{\gamma})$, it follows that $y \geq |x|/\sqrt{(\cosh R)^2 - 1}$.

From now on, we shall always parametrize $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ by arc-length, denoted by $\gamma_1(t)$ and $\gamma_2(s)$ respectively. The explicit expressions for the corresponding segments that we concern will be given in the proof case by case.

Proof of Lemma 16. Note that in this case, the image $\alpha(\tilde{\gamma}) = \gamma_2$ can be either a straight line or a half-circle parallel to $\tilde{\gamma} = \gamma_1$. We treat these two cases separately.

Let $\gamma_1(t) = (0, e^t)$ and $\gamma_2(s) = (a, e^s)$ parametrize the two unit geodesic segments respectively, where $a \in \mathbb{R}$, $t \in [0, 1]$ and s is in some unit closed interval of $\mathbb{R}$. See Figure 1.

The distance function is

$$\phi(t, s) = \mathrm{dist}(\gamma_1(t), \gamma_2(s)) = \mathrm{arcosh}\Big(1 + \frac{a^2 + (e^s - e^t)^2}{2e^{s+t}}\Big) = \mathrm{arcosh}\Big(\frac{a^2 + e^{2t} + e^{2s}}{2e^{t+s}}\Big).$$

Then we have

$$\phi''_{st}(t, s) = \frac{-8e^{2s+2t}a^2}{((a^2 + e^{2t} + e^{2s})^2 - 4e^{2s+2t})^{3/2}}.$$

One can obtain this expression by direct computations. See also the proof of (2.3.16) below.

By (2.3.6), we have $\phi \leq T$. Thus

$$a^2 e^{-t-s} + e^{t-s} + e^{s-t} \leq 2\cosh T,$$

which gives $s \in [-T, T+1]$ and $|a| \leq Ce^T$. Here C is independent of T.

To get the lower bound of $|\phi''_{st}|$, we need to use the condition that $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$. We claim that

$$\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})} \Rightarrow \ |a| \geq Ce^{-T}, \tag{2.3.13}$$

where C is independent of T. Note that if the segment $\{\gamma_2(s),\ s \in [-T, T+1]\}$ is completely included in $\mathrm{T}_R(\tilde{\gamma})$, then we must have $\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$, meaning that

$$e^{-T} \geq |a|\sqrt{(\cosh R)^2 - 1} \ \Rightarrow \ \alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})},$$

which implies our claim. Consequently,

$$|\phi''_{st}| \geq C\frac{e^{-2T}e^{-2T}}{e^{6T}} = Ce^{-10T}.$$

This gives the lower bound of $|\phi''_{st}|$. Moreover, direct computations give

$$\phi'''_{stt} = \frac{-16a^2e^{2s+2t}((a^2+e^{2s})^2 + e^{2s+2t} - a^2e^{2t} - 2e^{4t})}{((a^2+e^{2t}+e^{2s})^2 - 4e^{2s+2t})^{5/2}},$$

$$\phi''''_{sttt} = \frac{-32a^2e^{2s+2t}((a^2+e^{2s})^4 + \text{lower order terms})}{((a^2+e^{2t}+e^{2s})^2 - 4e^{2s+2t})^{7/2}}.$$

Here the lower order terms in the bracket are lower order as multivariate polynomials of a and e^s.

The upper bounds can be estimated similarly. By (2.3.6), we have $\phi \geq 2$, namely $a^2 + e^{2t} + e^{2s} \geq 2(\cosh 2)e^t e^s$. Thus

$$a^2 + e^{2t} + e^{2s} - 2e^{s+t} \geq (2\cosh 2 - 2)e^t e^s \geq Ce^{-T}.$$

So we have

$$(a^2 + e^{2t} + e^{2s})^2 - 4e^{2s+2t} \geq Ce^{-2T}.$$

Thus

$$|\phi''_{st}| \leq C\frac{e^{2T}e^{2T}}{e^{-3T}} \leq Ce^{7T}.$$

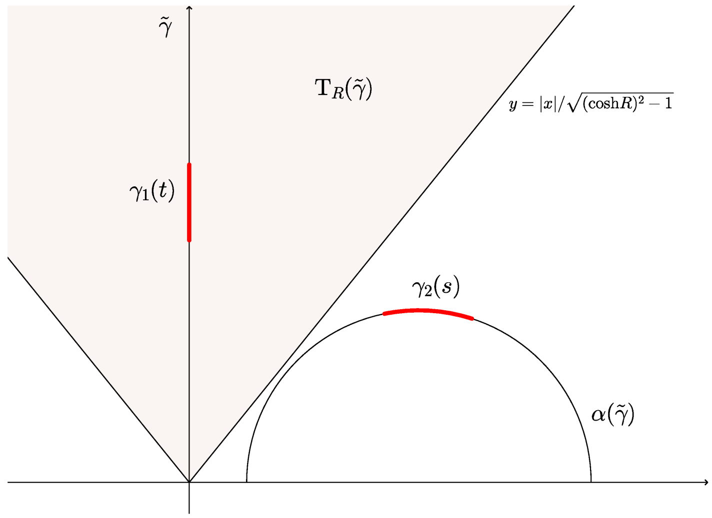

Figure 2.2: $\alpha(\tilde{\gamma})$ is a half-circle parallel to $\tilde{\gamma}$.

$$|\phi'''_{stt}| \le C\frac{e^{2T}e^{2T}e^{4T}}{e^{-5T}} \le Ce^{13T},$$

and

$$|\phi''''_{sttt}| \le C\frac{e^{2T}e^{2T}e^{8T}}{e^{-7T}} \le Ce^{19T}.$$

This completes the proof of the first case.

Now we turn to the case when γ_2 is a half-circle centered at $(a, 0)$ with radius $r > 0$. See Figure 2. Let $\gamma_1(t) = (0, e^t)$ and $\gamma_2(s) = (a + r\frac{1-e^{2s}}{1+e^{2s}}, \frac{2re^s}{1+e^{2s}})$ parametrize the two unit geodesic segments respectively, where $|a| \ge r > 0$, $, t \in [0, 1]$ and s is in some unit closed interval of $\mathbb{R}$. Without loss of generality, we may only consider the case $a \ge r > 0$. Then the distance function is

$$\phi(t, s) = \text{dist}(\gamma_1(t), \gamma_2(s)) = \text{arcosh}\Big(\frac{A}{4re^{s+t}}\Big), \tag{2.3.14}$$

where

$$A = e^{2s+2t} + (a - r)^2 e^{2s} + e^{2t} + (a + r)^2. \tag{2.3.15}$$

Thus we have

$$\phi''_{st} = \frac{16re^{2s+2t}(a + r + (a - r)e^{2s})(a^2 - r^2 + e^{2t})}{(A^2 - 16r^2 e^{2s+2t})^{3/2}}. \tag{2.3.16}$$

To see this, we write

$$e^{s+t}\cosh\phi = \frac{A}{4r}.$$

Taking derivatives on both sides, we obtain

$$(\phi_t' + \phi_s' + \phi_{ts}^{''})\text{sinh}\phi + (1 + \phi_t'\phi_s')\text{cosh}\phi = e^{s+t}/r. \tag{2.3.17}$$

Denote $X = e^{s+t}$, $Y = (a-r)^2 e^{s-t}$, $Z = e^{t-s}$, and $W = (a+r)^2 e^{-s-t}$. Since

$$4r\text{cosh}\phi = X + Y + Z + W,$$

taking derivatives gives

$$4r\phi_t'\text{sinh}\phi = X - Y + Z - W, \quad 4r\phi_s'\text{sinh}\phi = X + Y - Z - W.$$

Then we multiply both sides of (2.3.17) by $4r^2(\text{sinh}\phi)^2$ and use the hyperbolic trigonometric identity $(\text{sinh}\phi)^2 = (\text{cosh}\phi)^2 - 1$ to obtain

$$4r^2(\text{sinh}\phi)^3\phi_{st}^{''} = (a-r)(X+W) + (a+r)(Y+Z) = e^{-s-t}(a+r+(a-r)e^{2s})(a^2 - r^2 + e^{2t}).$$

This gives our desired expression (2.3.16).

Again by (2.3.6), we get $\phi \leq T$. Namely,

$$(e^{2t} + (a-r)^2)e^{2s} - 4r(\text{cosh}T)e^t e^s + e^{2t} + (a+r)^2 \leq 0, \tag{2.3.18}$$

which implies

$$\frac{r}{4\text{cosh}T} \leq e^s \leq 4r\text{cosh}T. \tag{2.3.19}$$

Moreover, note that if we view the left hand side of (2.3.18) as a quadratic polynomial of e^s, then the discriminant has to be nonnegative:

$$16r^2(\text{cosh}T)^2 e^{2t} - 4(e^{2t} + (a-r)^2)(e^{2t} + (a+r)^2) \geq 0,$$

we obtain that

$$\frac{a}{r} \leq 2e\text{cosh}T, \tag{2.3.20}$$

$$|a-r| \leq 2e\text{cosh}T, \tag{2.3.21}$$

and

$$r \geq \frac{1}{2\cosh T}. \tag{2.3.22}$$

To get the lower bound of $|\phi''_{st}|$, we need to use the condition that $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$.

We claim that there exists some constant C independent of T such that

$$\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})} \Rightarrow r \leq C\cosh T \text{ or } |a-r| \geq \frac{1}{C\cosh T}. \tag{2.3.23}$$

Indeed, we shall prove the contrapositive:

$$r \geq C\cosh T \text{ and } |a-r| \leq \frac{1}{C\cosh T} \Rightarrow \alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}. \tag{2.3.24}$$

We obtain this by showing that under the above assumptions on r and $|a-r|$, the segment $\{\gamma_2(s), s \in [-\ln(4r^{-1}\cosh T), \ln(4r\cosh T)]\}$ is completely contained in $\mathrm{T}_R(\tilde{\gamma})$, which implies $\alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$.

By solving the polynomial system

$$\begin{cases} y = |x|/\sqrt{(\cosh R)^2 - 1} \\ (x-a)^2 + y^2 = r^2 \end{cases}$$

and recalling that

$$x = a + r\frac{1-e^{2s}}{1+e^{2s}},$$

we can see that

$$\begin{aligned} &\{\gamma_2(s) : s \in \mathbb{R}\} \cap \mathrm{T}_R(\tilde{\gamma}) \\ &= \{\gamma_2(s) : (a-r)^2 e^{4s} + 2(a^2 - (2(\cosh R)^2 - 1)r^2)e^{2s} + (a+r)^2 \leq 0\}. \end{aligned} \tag{2.3.25}$$

Note that our assumptions imply $a/r \leq \cosh R$, namely

$$\begin{cases} r \geq C\cosh T \\ |a-r| \leq (C\cosh T)^{-1} \end{cases} \Rightarrow a/r \leq 1 + (C\cosh T)^{-2} \leq \cosh R.$$

Thus in the case when $a \neq r$, the RHS of (2.3.25) becomes

$$\{\gamma_2(s) : u_- \leq e^{2s} \leq u_+\}, \tag{2.3.26}$$

where

$$u_{\pm} = \frac{(2(\cosh R)^2 - 1) - (a/r)^2 \pm \sqrt{((\cosh R)^2 - (a/r)^2)((\cosh R)^2 - 1)}}{(a/r - 1)^2}. \tag{2.3.27}$$

It is easy to see that

$$u_{-} \le \frac{((a/r)^2 - 1)^2}{(a/r-1)^2(2(\cosh R)^2 - 1 - (a/r)^2)} \le \frac{(a/r+1)^2}{(\cosh R)^2 - 1} \le \frac{\cosh R + 1}{\cosh R - 1}, \tag{2.3.28}$$

$$u_{+} \ge \frac{(2(\cosh R)^2 - 1) - (a/r)^2}{(a/r-1)^2} \ge \frac{(\cosh R)^2 - 1}{(a/r-1)^2}. \tag{2.3.29}$$

So under our assumptions, if we choose $C = 4\sqrt{\cosh R + 1}/\sqrt{\cosh R - 1}$, we see that

$$a \ne r \text{ and } \begin{cases} r \ge C\cosh T \\ |a - r| \le (C\cosh T)^{-1} \end{cases} \Rightarrow \begin{cases} u_{-} \le r^2(4\cosh T)^{-2} \\ u+ \ge (4r\cosh T)^2 \end{cases} \Rightarrow \alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}. \tag{2.3.30}$$

In the easier case $a = r$, we have $u_{+} = +\infty$. Consequently, we obtain

$$\begin{cases} r \ge C\cosh T \\ |a - r| \le (C\cosh T)^{-1} \end{cases} \Rightarrow \alpha \in \Gamma_{\mathrm{T}_R(\tilde{\gamma})}.$$

This finishes the proof of our claim.

We note that by $\phi \le T$,

$$|\phi_{st}^{''}| \ge |\phi_{st}^{''}| \left(\frac{A}{4re^{s+t}\cosh T}\right)^2 \ge \frac{|a + r + (a-r)e^{2s}||a^2 - r^2 + e^{2t}|}{(\cosh T)^2 rA}. \tag{2.3.31}$$

Remark 3. Since we have not used the assumption that $a \ge r$ so far, (2.3.14)-(2.3.31) are applicable later to the case $a < r$ in the proof of Lemma 17.

We proceed by estimating $|\phi_{st}^{''}|$ for the two cases in (2.3.23) separately. By (2.3.31), it suffices to obtain a good lower bound for the numerator of the right hand side.

(I) Assume $r \le C\cosh T$.

If $a - r \ge 1$, then by (2.3.19)-(2.3.22), A is bounded by $Cr^2(a-r)^2(\cosh T)^2$. And the numerator in the right hand side of (2.3.31) is bounded below by $(a-r)e^{2s}(a^2 - r^2)$. Using (2.3.19), we get

$$|\phi_{st}^{''}| \ge \frac{C}{(\cosh T)^2} \frac{(a+r)(a-r)^2 r^2 (\cosh T)^{-2}}{r(r^2(a-r)^2(\cosh T)^2)} \ge Ce^{-6T}.$$

If $0 \leq a - r \leq 1$, then similarly we have

$$|\phi_{st}^{''}| \geq \frac{C}{(\cosh T)^2} \frac{a+r}{r(r^2(\cosh T)^2)} \geq Ce^{-6T}.$$

(II) Assume $a - r \geq \frac{1}{C\cosh T}$. We may assume further that $r \geq 1$, otherwise it is reduced to the first case.

If $a - r \geq 1$, then using (2.3.19)-(2.3.22) we get

$$|\phi_{st}^{''}| \geq \frac{C}{(\cosh T)^2} \frac{(a+r)(a-r)^2 r^2 (\cosh T)^{-2}}{r((a-r)^2 r^2 (\cosh T)^2)} \geq Ce^{-6T}.$$

If $0 \leq a - r \leq 1$ then similarly we obtain

$$|\phi_{st}^{''}| \geq \frac{C}{(\cosh T)^2} \frac{(a+r)(a-r)^2 r^2 (\cosh T)^{-2}}{r(r^2(\cosh T)^2)} \geq Ce^{-8T}.$$

Note that the constant C is independent of T. Hence we finish the proof of the lower bound of $|\phi_{st}^{''}|$.

The upper bounds can be obtained in a similar fashion. By the expression (2.3.16), direct computations give

$$\phi_{stt}^{'''} = \frac{-32re^{2s+2t}(a+r+(a-r)e^{2s})((a+r)(a-r)^5 e^{4s} + \text{lower order terms})}{(A^2 - 16r^2 e^{2s+2t})^{5/2}}, \tag{2.3.32}$$

$$\phi_{sttt}^{''''} = \frac{-64re^{2s+2t}(a+r+(a-r)e^{2s})((a+r)(a-r)^9 e^{8s} + \text{lower order terms})}{(A^2 - 16r^2 e^{2s+2t})^{7/2}}. \tag{2.3.33}$$

Here again the lower order terms in the bracket are lower order as multivariate polynomials in terms of a, r and e^s.

By (2.3.32)-(2.3.33), we only need to estimate the lower bound of $A^2 - 16r^2e^{2s+2t}$ and the upper bounds of the absolute values of the numerators. By (2.3.6), we have $\phi \geq 2$, namely $A \geq 4(\cosh 2)re^{s+t}$. Thus

$$A - 4re^{s+t} \geq (4\cosh 2 - 4)re^t e^s. \tag{2.3.34}$$

(I) Assume $r \leq C\cosh T$. Using (2.3.19)-(2.3.22) and (2.3.34), we get

$$A - 4re^{s+t} \geq (4\cosh 2 - 4)re^t e^s \geq C(\cosh T)^{-3},$$

which implies

$$A^2 - 16r^2 e^{2s+2t} \geq C(\cosh T)^{-6}.$$

Then by (2.3.19)-(2.3.21),

$$|\phi_{st}^{''}| \leq \frac{C(\cosh T)(\cosh T)^4(\cosh T)^5(\cosh T)^2}{(\cosh T)^{-9}} \leq Ce^{21T},$$

$$|\phi_{stt}^{'''}| \leq \frac{C(\cosh T)(\cosh T)^4(\cosh T)^5(\cosh T)^{14}}{(\cosh T)^{-15}} \leq Ce^{39T},$$

$$|\phi_{stt}^{'''}| \leq \frac{C(\cosh T)(\cosh T)^4(\cosh T)^5(\cosh T)^{26}}{(\cosh T)^{-21}} \leq Ce^{57T}.$$

(II) Assume $r \geq C\cosh T$. By (2.3.19) and (2.3.34), we have

$$A - 4re^{s+t} \geq (4\cosh 2 - 4)re^t e^s \geq Cr^2(\cosh T)^{-1},$$

which implies

$$A^2 - 16r^2e^{2s+2t} \geq Cr^4(\cosh T)^{-2}.$$

Thus by (2.3.19)-(2.3.21),

$$|\phi_{st}^{''}| \leq \frac{Cr^3(\cosh T)^2(r^2(\cosh T)^3)(r\cosh T)}{((\cosh T)^{-2}r^4)^{3/2}} \leq Ce^{9T},$$

$$|\phi_{stt}^{'''}| \leq \frac{Cr^3(\cosh T)^2(r^2(\cosh T)^3)(r^5(\cosh T)^9)}{((\cosh T)^{-2}r^4)^{5/2}} \leq Ce^{19T},$$

$$|\phi_{sttt}^{''''}| \leq \frac{Cr^3(\cosh T)^2(r^2(\cosh T)^3)(r^9(\cosh T)^{17})}{((\cosh T)^{-2}r^4)^{7/2}} \leq Ce^{29T}.$$

Since the constant C is independent of T, the proof is complete.

Remark 4. Since we did not use the assumption that $a \geq r$ in the proof of the upper bounds of various derivatives, these upper bounds are also valid for the case $a < r$ in Lemma 17. Indeed, the upper bounds for the derivatives hold for not only $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$ but all $\alpha \neq Id$, as we only use the condition that $2 \leq \phi \leq T$.

$\square$

Proof of Lemma 17. Let $\gamma_1(t) = (0, e^t)$ and $\gamma_2(s) = (a + r\frac{1-e^{2s}}{1+e^{2s}}, \frac{2re^s}{1+e^{2s}})$ parametrize the two unit geodesic segments respectively, where $r > |a| \geq 0$, $t \in [0,1]$ and s is in some unit closed interval of $\mathbb{R}$. Without loss of generality, we may only consider the case $r > a \geq 0$. The expressions of the distance function ϕ and its derivatives are the same as in (2.3.14)-(2.3.16), (2.3.32) and (2.3.33). Moreover,

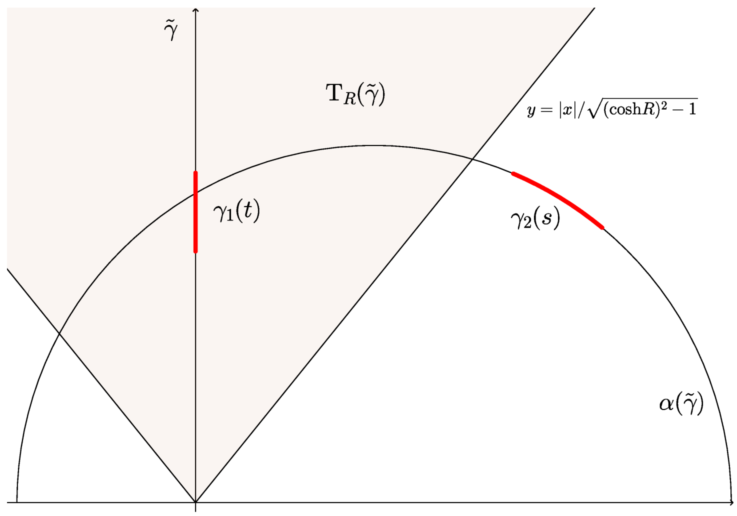

Figure 2.3: $t_0 \in [-1, 2]$

from Remark 3 we can see that (2.3.19)-(2.3.31) are also applicable. So the zero set of $\phi''_{st}(t, s)$ is

$$\Big\{(t, s) \in \mathbb{R}^2 : t = t_0 \text{ or } s = s_0, \text{ where } e^{2t_0} = r^2 - a^2 \text{ and } e^{2s_0} = \frac{r + a}{r - a}\Big\}.$$

It is not difficult to check that the point $p = \gamma_1(t_0) = \gamma_2(s_0)$ is the intersection point of the two infinite geodesics γ_1 and γ_2. If the unit geodesic segment $\gamma_2(s)$ passes through the intersection point, then this segment must be contained in $\mathrm{T}_R(\tilde{\gamma})$, thus our geodesic segment $\gamma_2(s)$ cannot pass through the point p. We need to consider the following two cases: (I) $t_0 \in [-1, 2]$, (II) $t_0 \notin [-1, 2]$. By Remark 4, we only need to consider the lower bounds of $|\phi''_{st}|$ or $|\phi''_{st}/(t - t_0)|$.

(I) Assume $t_0 \in [-1, 2]$. See Figure 2.3.

By Remark 3, the claim (2.3.23) is vaild. Since $t_0 \in [-1, 2]$, we have

$$e^{2t_0} = (r + a)(r - a) \in [e^{-2}, e^4], \tag{2.3.35}$$

which implies that the two conditions in (2.3.23), $r \leq C\cosh T$ and $|r - a| \geq \frac{1}{C\cosh T}$, are equivalent.

By (2.3.26), $\alpha \notin \Gamma_{\mathrm{T}_R(\tilde{\gamma})}$ implies $e^{2s} > u_+$ or $e^{2s} < u_-$. If $e^{2s} > u_+$, by (2.3.29) we have

$$(r - a)e^{2s} - (r + a) \geq \frac{(\cosh R)^2 - 1}{(1 - a/r)^2}(r - a) - (r + a)$$
$$= \frac{(\cosh R)^2 - 2 + (a/r)^2}{1 - a/r} r \geq ((\cosh R)^2 - 2)r.$$

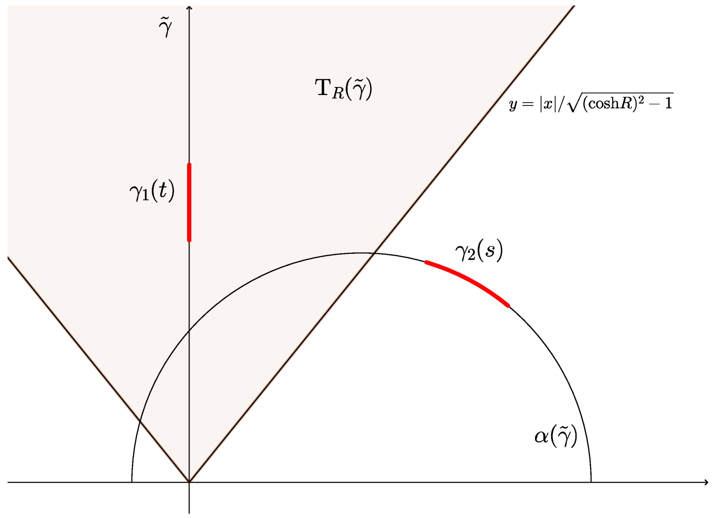

Figure 2.4: $t_0 \notin [-1, 2]$

If $e^{2s} < u_-$, similarly by (2.3.28), we have

$$\begin{aligned}(r+a) - (r-a)e^{2s} &\geq (r+a) - \frac{(a/r+1)^2}{(\cosh R)^2 - 1}(r-a) \\ &= \frac{(\cosh R)^2 - 2 + (a/r)^2}{(\cosh R)^2 - 1}(1 + a/r)r \geq \frac{(\cosh R)^2 - 2}{(\cosh R)^2 - 1} r.\end{aligned}$$

Hence for some constant C independent of T, we always have

$$|a + r + (a-r)e^{2s}| \geq Cr. \tag{2.3.36}$$

By (2.3.35) and (2.3.23), we see that $e^{-1} \leq r \leq C\cosh T$ and $|r - a| \leq e^2$. We get

$$A \leq Cr^2(\cosh T)^2.$$

Thus by (2.3.31), for $t \in [-1, 2] \setminus \{t_0\}$

$$\left|\frac{\phi''_{st}}{t - t_0}\right| \geq \frac{Cr}{(\cosh T)^2 r (r^2(\cosh T)^2)} \left|\frac{e^{2t} - r^2 + a^2}{t - t_0}\right| \geq Ce^{-6T},$$

where we used the fact $e^{2t_0} = r^2 - a^2$ and the mean value theorem. This lower bound also implies $\phi'''_{stt}(t_0, s) \neq 0$, and $\phi''_{ts} \neq 0$ for $t \in [-1, 2] \setminus \{t_0\}$.

(II) Assume $t_0 \notin [-1, 2]$. See Figure 4.

By Remark 3, the claim (2.3.23) is also applicable here. By (2.3.21), we have

$$A \leq Cr^2(\text{cosh}T)^4.$$

Since $t_0 \notin [-1, 2]$, $|e^{2t} - r^2 + a^2| = |e^{2t} - e^{2t_0}| \geq 1 - e^{-2}$. If $r \leq C(\text{cosh}T)^4$, by (2.3.31) and (2.3.36), we have

$$|\phi_{st}^{''}| \geq \frac{Cr}{(\text{cosh}T)^2 r(r^2(\text{cosh}T)^4)} \geq Ce^{-14T}.$$

If $|r - a| \geq \frac{1}{C\text{cosh}T}$ and $r \geq (\text{cosh}T)^4$, then

$$(r-a)e^{2s} - (a+r) \geq Cr^2(\text{cosh}T)^{-3} - 2r \geq Cr^2(\text{cosh}T)^{-3},$$

$$r^2 - a^2 - e^{2t} \geq Cr(\text{cosh}T)^{-1} - e^2 \geq Cr(\text{cosh}T)^{-1}.$$

Thus by (2.3.31) we get

$$|\phi_{st}^{''}| \geq \frac{C(r^2(\text{cosh}T)^{-3})(r(\text{cosh}T)^{-1})}{(\text{cosh}T)^2 r(r^2(\text{cosh}T)^4)} \geq Ce^{-10T},$$

which completes our proof. □

Remark 5. As pointed out in [4], the various upper bounds for pure derivatives $|D_t^\alpha \phi| + |D_s^\alpha \phi| \leq C_\alpha e^{CT}$ follow from Proposition 3 and Lemma 4 in [2]. But it seems that the upper bounds for mixed derivatives are unknown. So we are including the proofs for these upper bounds for the sake of completeness.

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Curriculum Vitae

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